

# Cyclic dominance in a 2-person Rock-Scissors-Paper game

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## Abstract

The Rock-Scissors-Paper game has been studied in the context of single population dynamics to account for cyclic behaviour. We use a 2-person parametrised version of this game to illustrate how cyclic behaviour is still a dominant feature of the dynamics. The cyclic behaviour is observed near a heteroclinic cycle with two nodes such that, at each node, players alternate in winning and losing. This cycle is shown to be as stable as possible for a wide range of parameter values. The parameters are related to the players' payoff when a tie occurs. This cycle is part of a heteroclinic network: there are two other cycles with two nodes and two cycles with three nodes. The cycles with two nodes exhibit some intermediate stability for a large subset of parameter space, contained in the complement of the set of stability of the first cycle. These two-node cycles represent oscillations between a tie between players and a win for only one of the players. The three-node cycles are always unstable and describe a cycle among all possible outcomes in the game, in the two possible sequences. Using some applications to price setting models, we propose what could be a starting point for the contribution of the Rock-Scissors-Paper game to the understanding of cyclic dominance in two-player games.

**JEL codes:** C72, C73, C02

# 1 Introduction

The Rock-Scissors-Paper game (henceforth, RSP) has been used to model behaviour and learning in both economics and the life sciences. In the context of a 1-player or population dynamics, it provides a good model for both convergent and oscillating dynamics, the type of dynamics depending, of course, on some parameters of the model. In the life sciences, a classic example is that of the evolution of the three types in a population of lizards of the species *Uta Stansburiana*. The model of RSP replicates well the oscillatory nature among the three types. See, for instance, Sinervo and Lively [29]<sup>1</sup> for the story of the lizards or Szolnoki *et al.* [30] for a general review of cyclic behaviour in nature. In economics, the mixed-strategy Nash equilibrium has been used to illustrate the price dispersion and the game itself to support the cyclic behaviour of prices (set by a population of sellers or firms). The literature goes back at least to Edgeworth [8], who predicted the existence of price cycles without mention of RSP. From the point of view of applications, price cycles were observed by Noel [22] in the Toronto retail gasoline market, where cycling among several price levels, from low to high and back again, is reported for major and independent firms. The use of the RSP game to model price dispersion can be found in Hopkins and Seymour [14], where the existence of price dispersion seems to depend on the absence of informed consumers. In laboratory experiments price cycling is reported in Cason *et al.* [5] and an experiment directed at illustrating cyclic behaviour in RSP can be found in Cason *et al.* [4], where the issue of learning is present.

Learning is a natural environment for the use of the RSP game as it appears in the form of replicator dynamics. In [14], learning on the side of consumers prevents price dispersion. Consumers' choice of brands is mentioned by Börgers and Sarin [2] as a possible application of reinforcement learning. The model of [2] is further developed in Lahkar and Seymour [18] to include the RSP game exhibiting local convergence to the interior rest point.

Even though sometimes the seller perspective is in focus and other times the buyers point of view is taken, the applications are so far those of a single population of individuals rather than the strategic interaction of two (or more) individuals or populations. An exception to this is the work by Friedman [10] although his model is not by replicator dynamics. From a strictly abstract point of view, the interaction of individuals modelled by the RSP game has been addressed numerically by Sato *et al.* [26] and, building

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<sup>1</sup>The references given here are merely illustrative, do not aim at being comprehensive and depend on the authors' personal library.

from this, analytically by Aguiar and Castro<sup>2</sup> [1].

In this paper, we address stability of cyclic behaviour in the strategic interaction of two players of RSP. Aguiar and Castro [1] show that the dynamics exhibit a heteroclinic network made of pairs of pure strategies and solutions connecting them in a suitable quotient space. This network can be seen as consisting of

- (a) three cycles, one of which involves alternate win-loss of both players, and two involving a tie and loss by only one of the players;
- (b) two cycles for which play goes through all possible combinations of outcomes in the two possible orders.

We present a comprehensive study of the stability of the cycles in the RSP game, showing that this stability depends on the payoffs received for a tie, which we allow to vary in a range that makes a tie almost (but not quite) as good as a win or almost (but not quite) as bad as a loss. Although cycles in a network cannot be asymptotically stable, they can exhibit a strong type of stability, known as *essential asymptotic stability*, first introduced by Melbourne [20]. We show that the cycle where players never tie is essentially asymptotically stable when the sum of payoffs for a tie is negative. In this case, a tie is not an attractive outcome for at least one of the players for whom the payoff is negative (the payoffs for winning and losing are normalised to +1 and -1, respectively) and hence, as an outcome, it is avoided.

Our model allows for further development of strategic interaction in economics. The dynamics of the gasoline retail market in Noel [22], show that the major and independent firms alternate in setting the highest price (see Figure 1 therein). If we assume that consumers buy at the lowest price, then major and independent firms alternate in winning and losing in an RSP game where the actions are “fix lowest price”, “fix highest price” and “fix equal price”. This corresponds to the win-loss cycle in (a) above. Considering the pricing model of Hopkins and Seymour [14], our RSP allows the introduction of the consumers as players by choosing to be “uninformed”, “little-informed” and “well-informed”. We discuss these models further in the final section.

Admittedly, the dynamics arising from the interaction of two players can be very complex and we do not attempt to find detailed specific applications. We do however hope that this first approach and its results can open the door to further research in this context.

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<sup>2</sup>A mistake in their Lemma 4.9, recently corrected by the authors in [11], leads to the statement that no cyclic behaviour is very stable.

This article is organised as follows: the next section contains preliminary material which may be skipped by the reader familiar with the dynamics near heteroclinic networks. Section 3 describes the 2-person RSP game and its cycles. Section 4 contains the necessary information to describe the trajectories of points near each cycle in the RSP network. This is then used in Section 5 to provide a thorough study of the stability of all the cycles in the dynamics of RSP. Detailed calculations are deferred to an appendix. Section 6 suggests a possible first approach to an extension to two players of the models by Noel [22] and by Hopkins and Seymour [14]. The last section concludes.

## 2 Definitions and preliminaries

Consider a continuous-time dynamical system  $\dot{x} = f(x)$ , where  $x \in \mathbb{R}^n$  and  $f : \mathbb{R}^n \rightarrow \mathbb{R}^n$  is a smooth vector field. We say that  $f$  is  $\Gamma$ -equivariant for some finite group  $\Gamma$  acting orthogonally on  $\mathbb{R}^n$  if  $f(\gamma \cdot x) = \gamma \cdot f(x)$ , for all  $\gamma \in \Gamma$  and  $x \in \mathbb{R}^n$ . Then we say that  $\gamma \in \Gamma$  is a symmetry of  $f$ . Background on differential equations with symmetry can be found in [12].

The  $\Gamma$ -orbit of  $x \in \mathbb{R}^n$  is the set

$$\Gamma(x) = \{\gamma \cdot x, x \in \Gamma\}.$$

In particular, the elements in the  $\Gamma$ -orbit of an equilibrium for the flow of  $f$  are also equilibria. A group orbit of equilibria is called a relative equilibrium. Given equilibria  $\xi_i, \xi_j \in \mathbb{R}^n$ , if there is a connecting trajectory from  $\xi_i$  to  $\xi_j$ , denoted by  $[\xi_i \rightarrow \xi_j]$ , then its image by the action of  $\gamma \in \Gamma$  is a connection  $[\gamma \cdot \xi_i \rightarrow \gamma \cdot \xi_j]$ .

The set of all  $\Gamma$ -orbit of a subset  $S \subset \mathbb{R}^n$  is called the quotient space or orbit space. If  $S$  is invariant under the flow of  $f$ , then the flow of  $f$  restricts to a flow on the quotient space so that  $S$  can be reduced to the quotient space by identifying points in the same group orbit.

A heteroclinic cycle is an invariant set  $X \subset \mathbb{R}^n$  which consists of an ordered set of hyperbolic equilibria  $\{\xi_1, \dots, \xi_m\}$  for the flow of  $f$  and trajectories  $[\xi_j \rightarrow \xi_{j+1}] \subset W^u(\xi_j) \cap W^s(\xi_{j+1}) \neq \emptyset$ ,  $j = 1, \dots, m$ , connecting them, where  $\xi_{m+1} = \xi_1$ , and  $W^u$  and  $W^s$  are unstable and stable manifolds, respectively. A heteroclinic network is a connected union of finitely many heteroclinic cycles.

In general a connection between two saddles can be broken by arbitrarily small perturbations of  $f$ , which makes the cycle structurally unstable. A sufficient condition for preserving its structure relies on the existence of spaces invariant under the flow of  $f$  where connections are of saddle-sink

type. Such a heteroclinic cycle is called robust. Robust heteroclinic cycles arise naturally in equivariant systems (e.g. [9]) as well as in game theory and population dynamics (e.g. [13]). In equivariant systems spaces of points fixed by the group action are naturally flow invariant. In the context of population dynamics such flow invariant spaces arise in the form of extinction hyperplanes.

Assume that for each  $j = 1, \dots, m$  there exists a flow invariant subspace  $P_j$  such that  $\xi_j$  is a hyperbolic saddle and  $\xi_{j+1}$  is a hyperbolic sink in  $P_j$ , and  $[\xi_j \rightarrow \xi_{j+1}] \subset P_j$ . Notice that  $P_j$  has neither to be linear nor the smallest possible subspace. Even so, we take  $P_j$  as the smallest linear subspace. Set  $L_j = P_{j-1} \cap P_j$  with  $P_0 = P_m$ . Obviously,  $\xi_j \in L_j$ .

Criteria for asymptotic and non-asymptotic stability of a heteroclinic cycle can be given in terms of the eigenvalues of the Jacobian matrix associated with the linearization of  $f$  about each equilibrium. According to the geometry of the eigenspaces at  $\xi_j$ , eigenvalues are divided into four classes, where  $P \ominus L$  denotes the orthogonal complement in  $P$  of the subspace  $L$ :

- radial eigenvalues  $(-\mathbf{r}_j)$  with associated eigenvectors in  $L_j$ ;
- contracting eigenvalues  $(-\mathbf{c}_j)$  with associated eigenvectors in  $V_j(\mathbf{c}) = P_{j-1} \ominus L_j$ ;
- expanding eigenvalues  $(\mathbf{e}_j)$  with associated eigenvectors in  $V_j(\mathbf{e}) = P_j \ominus L_j$ ;
- transverse eigenvalues  $(\mathbf{t}_j)$  with associated eigenvectors in  $V_j(\mathbf{t}) = (P_{j-1} + P_j)^\perp$ .

We set  $-\mathbf{r}_j = \{-r_{j,l}\}$  for  $1 \leq l \leq n_{r,j}$ ,  $-\mathbf{c}_j = \{-c_{j,l}\}$  for  $1 \leq l \leq n_{c,j}$ ,  $-\mathbf{e}_j = \{-e_{j,l}\}$  for  $1 \leq l \leq n_{e,j}$  and  $-\mathbf{t}_j = \{-t_{j,l}\}$  for  $1 \leq l \leq n_{t,j}$ , such that  $n = n_{r,j} + n_{c,j} + n_{e,j} + n_{t,j}$ . We focus on the case in which eigenvalues are all real. The numbers  $r_{j,l}$ ,  $c_{j,l}$  and  $e_{j,l}$  are positive but  $t_{j,l}$  can be either positive or negative. Before a heteroclinic network, the type of the eigenvalue at each equilibrium is determined by the cycle taken into account. For example, an eigenvalue may be contracting with respect to one cycle and transverse with respect to another cycle.

The issue of stability for robust heteroclinic cycles has been a subject of interest. Prominent results for asymptotic stability were established by Krupa and Melbourne [16, 17]. Nevertheless, numerical simulations reveal that a heteroclinic cycle may have strong attractivity properties even though it is not asymptotically stable. Then several forms of non-asymptotic stability have been employed as essentially asymptotic stability and fragmentarily

asymptotic stability. This is particularly appropriate for cycles put together in a heteroclinic network since none of them can be asymptotically stable. In fact an equilibrium common to more than one cycle always admits an unstable direction with respect to one of the cycles to which it belongs.

Let  $X \subset \mathbb{R}^n$  be a compact set invariant under the flow,  $\Phi_t(\cdot)$ , associated to  $f$  and  $\epsilon, \delta > 0$ . Following the terminology of [24], we denote by  $B_\epsilon(X)$  an  $\epsilon$ -neighbourhood of  $X$ . We write  $\mathcal{B}_\delta(X)$  for the  $\delta$ -local basin of attraction of  $X$ ,

$$\mathcal{B}_\delta(X) := \{x \in \mathbb{R}^n : \omega(x) \subset X \text{ and } \Phi_t(x) \in B_\delta(X) \ \forall t > 0\},$$

where  $\omega(x)$  is the  $\omega$ -limit of  $x$ . By  $\ell(\cdot)$  we denote Lebesgue measure.

Brannath [3] introduces the concept of essential asymptotic stability (e.a.s.) which is the strongest intermediate form of stability. Roughly speaking, an e.a.s. invariant object attracts all nearby trajectories except for a cuspidal region of points sufficiently thin for making it visible in experiments.

**Definition 2.1** (Definition 1.2 in [3]). *A compact invariant set  $X$  is essentially asymptotically stable (e.a.s.) if it is asymptotically stable relative to a set  $N \subset \mathbb{R}^n$  which satisfies*

$$\lim_{\epsilon \rightarrow 0} \frac{\ell(B_\epsilon(X) \cap N)}{\ell(B_\epsilon(X))} = 0.$$

A form of weak attractiveness comes up in Podvigina [23] through the definition of fragmentary asymptotic stability (f.a.s.).

**Definition 2.2** (Definition 2 in [23]). *A compact invariant set  $X$  is fragmentarily asymptotically stable (f.a.s.) if for any  $\delta > 0$*

$$\ell(\mathcal{B}_\delta(X)) > 0.$$

Podvigina and Ashwin [24] define a stability index in order to quantify locally the size of the basin of attraction for any compact invariant set  $X \subset \mathbb{R}^n$ . It provides a useful tool to describe stability properties, either at global or at local level, by adapting properly to the type of the basin of attraction of  $X$ . An important feature of the stability index is that it is constant along trajectories of the flow. Then a heteroclinic cycle is characterized through a set of stability indices of the connecting trajectories.

**Definition 2.3** (Definition 5 in [24]). *For a point  $x \in X$  and  $\epsilon, \delta > 0$ , define*

$$\Sigma_{\epsilon, \delta}(x) := \frac{\ell(B_\epsilon(x) \cap \mathcal{B}_\delta(X))}{\ell(B_\epsilon(x))}.$$

The local stability index of  $X$  at  $x$  is

$$\sigma_{loc}(x) := \sigma_{loc,+}(x) - \sigma_{loc,-}(x),$$

which exists when the following converge:

$$\sigma_{loc,-}(x) := \lim_{\delta \rightarrow 0} \lim_{\epsilon \rightarrow 0} \left[ \frac{\ln(\Sigma_{\delta,\epsilon}(x))}{\ln(\epsilon)} \right], \quad \sigma_{loc,+}(x) := \lim_{\delta \rightarrow 0} \lim_{\epsilon \rightarrow 0} \left[ \frac{\ln(1 - \Sigma_{\delta,\epsilon}(x))}{\ln(\epsilon)} \right].$$

For fixed  $\delta$ , we use the convention that  $\sigma_{loc,-}(x) = \infty$  if there is an  $\epsilon_0 > 0$  such that  $\Sigma_{\delta,\epsilon}(x) = 0$  for all  $\epsilon < \epsilon_0$ , and  $\sigma_{loc,+}(x) = \infty$  if  $\Sigma_{\delta,\epsilon}(x) = 1$  for all  $\epsilon < \epsilon_0$ . Note that  $\sigma_{loc,\pm}(x) \geq 0$  and so we can assume that  $\sigma_{loc}(x) \in [-\infty, \infty]$ .

The following two results state the relation between non-asymptotic stability of a heteroclinic cycle or network  $X \subset \mathbb{R}^n$  and the stability indices along its connections.

**Lemma 2.4** (Lemma 2.4 in [11]). *Suppose that for all  $x \in X$  the local stability index is defined. If there is a point  $x \in X$  such that  $-\infty < \sigma_{loc}(x)$  then  $X$  is f.a.s.*

Let  $\ell_1(\cdot)$  denote the 1-dimensional Lebesgue measure.

**Theorem 2.5** (Theorem 3.1 in [19]). *Let  $X \subset \mathbb{R}^n$  be a heteroclinic cycle or network with finitely many equilibria and connecting trajectories. Suppose that  $\ell_1(X) < \infty$  and that the local stability index  $\sigma_{loc}(x)$  exists and is not equal to zero for all  $x \in X$ . Then, generically, we have  $X$  is e.a.s. is equivalent to  $\sigma_{loc} > 0$  along all connecting trajectories.*

In what follows we drop the subscript *loc* for ease of notation.

### 3 The Rock-Scissors-Paper game

We base our description of the Rock-Scissors-Paper (RSP) game on [26] and [1]. Two agents,  $X$  and  $Y$ , simultaneously choose from one of three possible actions, R (rock), S (scissors) and P (paper), such that R beats S, S beats P, P beats R. The payoff of a winning action is +1 while the payoff of a losing action is -1. If the choice of action coincides, then a tie occurs and the payoffs are parametrized by quantities  $-1 < \varepsilon_x < 1$  and  $-1 < \varepsilon_y < 1$ , for agent  $X$  and for agent  $Y$ , respectively. The normal form representation of the game is given by two normalized payoff matrices

$$A = \begin{bmatrix} 0 & 1 - \varepsilon_x & -1 - \varepsilon_x \\ -1 - \varepsilon_x & 0 & 1 - \varepsilon_x \\ 1 - \varepsilon_x & -1 - \varepsilon_x & 0 \end{bmatrix}, \quad B = \begin{bmatrix} 0 & 1 - \varepsilon_y & -1 - \varepsilon_y \\ -1 - \varepsilon_y & 0 & 1 - \varepsilon_y \\ 1 - \varepsilon_y & -1 - \varepsilon_y & 0 \end{bmatrix},$$

whose order of columns and rows follows the actions R, S, P. Each element of the matrix  $A$  (resp.  $B$ ) is the payoff of the row agent  $X$  (resp.  $Y$ ) playing against the column agent  $Y$  (resp.  $X$ ).

Suppose that the game is repeated indefinitely. Every agent knows the payoffs and strategies available to its opponent and perfectly recalls the past movements at any stage. Suppose further that agents independently attempt to maximize their own payoffs over time by adapting their behaviours through reinforcement learning dynamics. In this sense, at each stage, the agents' choices are expressed as random in the form of state probabilities of playing actions R, S, P. For agents  $X$  and  $Y$ , these probabilities are  $\mathbf{x} = (x_1, x_2, x_3)$  and  $\mathbf{y} = (y_1, y_2, y_3)$ , respectively, with  $x_1, x_2, x_3 \geq 0$ ,  $x_1 + x_2 + x_3 = 1$  and  $y_1, y_2, y_3 \geq 0$ ,  $y_1 + y_2 + y_3 = 1$ . States  $\mathbf{x}$  and  $\mathbf{y}$  describe mixed strategies and the set of all such mixed strategies is a two-dimensional simplex, individually denoted by  $\Delta_X$  and  $\Delta_Y$ . In particular, the vertices of these simplices are the unit vectors  $(1, 0, 0)$ ,  $(0, 1, 0)$ ,  $(0, 0, 1)$  and correspond to pure strategies R, S, P.

After each stage, the agents update their states in response to the payoffs received in the past. Positive payoffs lead to reinforcing behaviours, increasing the probability of the action just chosen. On other hand, negative payoffs induce a reduction in the probability of the action currently in use. From here, the state of the game  $(\mathbf{x}, \mathbf{y})$  evolves according to the coupled replicator equations

$$\begin{aligned}\dot{x}_i &= x_i [(A\mathbf{y})_i - \mathbf{x}^T A\mathbf{y}], \quad i = 1, 2, 3 \\ \dot{y}_j &= y_j [(B\mathbf{x})_j - \mathbf{y}^T B\mathbf{x}], \quad j = 1, 2, 3\end{aligned}\tag{1}$$

on a four-dimensional collective state space  $\Delta = \Delta_X \times \Delta_Y$  contained in  $\mathbb{R}^6$ . The first equation determines the adjustment rule of the state or strategy of agent  $X$  where  $(A\mathbf{y})_i$  is the expected payoff of the pure strategy  $x_i = 1$  and  $\mathbf{x}^T A\mathbf{y}$  is the expected payoff of the strategy  $\mathbf{x}$ , given that agent  $Y$  plays the strategy  $\mathbf{y}$ . The second equation is similar regarding the agent  $Y$ .

The vector field (1) is symmetric under the finite group  $\Gamma$  generated by the action

$$(x_1, x_2, x_3; y_1, y_2, y_3) \mapsto (x_3, x_1, x_2; y_3, y_1, y_2).$$

Moreover, the coordinate hyperplanes are flow-invariant and the same holds for all sub-simplices of  $\Delta$ .

The dynamic equilibria of (1) satisfy  $\dot{x}_i = 0$  and  $\dot{y}_j = 0$ , for all  $i, j = 1, 2, 3$ . Then, all vertices  $(\mathbf{x}, \mathbf{y}) \in \Delta$ , where  $\mathbf{x}, \mathbf{y} \in \{R, S, P\}$ , are equilibria as well as  $(\mathbf{x}^*, \mathbf{y}^*) = (\frac{1}{3}, \frac{1}{3}, \frac{1}{3}; \frac{1}{3}, \frac{1}{3}, \frac{1}{3})$  which lies in the interior of  $\Delta$ . The latter is also the unique Nash equilibrium of the game.

The nine vertices are hyperbolic saddle points and, together with the



edges of  $\Delta$ , form a heteroclinic network. Aguiar and Castro [1] analitically prove the existence of the RSP network and establish two equivalent descriptions thereof obtained from the union of three heteroclinic cycles  $C_0, C_1, C_2$ , with

$$\begin{aligned} C_0 &= [(R, P) \rightarrow (S, P) \rightarrow (S, R) \rightarrow (P, R) \rightarrow (P, S) \rightarrow (R, S) \rightarrow (R, P)] \\ C_1 &= [(R, S) \rightarrow (R, R) \rightarrow (P, R) \rightarrow (P, P) \rightarrow (S, P) \rightarrow (S, S) \rightarrow (R, S)] \\ C_2 &= [(S, R) \rightarrow (R, R) \rightarrow (R, P) \rightarrow (P, P) \rightarrow (P, S) \rightarrow (S, S) \rightarrow (S, R)], \end{aligned}$$

and the union of two heteroclinic cycles  $C_3, C_4$ , with

$$\begin{aligned} C_3 &= [(R, S) \rightarrow (R, R) \rightarrow (R, P) \rightarrow (S, P) \rightarrow (S, S) \rightarrow (S, R) \rightarrow (P, R) \rightarrow \\ &\quad \rightarrow (P, P) \rightarrow (P, S) \rightarrow (R, S)] \\ C_4 &= [(S, R) \rightarrow (R, R) \rightarrow (P, R) \rightarrow (P, S) \rightarrow (S, S) \rightarrow (R, S) \rightarrow (R, P) \rightarrow \\ &\quad \rightarrow (P, P) \rightarrow (S, P) \rightarrow (S, R)]. \end{aligned}$$

We point out that all connections are one-dimensional and are contained in two-dimensional flow-invariant subspaces that are not vector subspaces. They are the product of a boundary face of an agent's state space with a vertex of the oponent's state space. This means that  $x_i = 0$  and  $y_j = 1$  or  $x_i = 1$  and  $y_j = 0$ , for some  $i, j = 1, 2, 3$ . Furthermore, within these subspaces, the heteroclinic connections are of saddle-sink type which guarantees the robustness of the cycles. Since it is more convenient to use vector spaces, we look at the connections inside a three-dimensional vector space, also invariant under the flow where the connections exist in a robust way.

Due to symmetry, the dynamics of the system can be studied from the dynamics on a quotient space. This reduces the complexity of dynamical objects, preserving their stability properties. In particular, equations (1) restricted to the quotient space give rise to a quotient heteroclinic network made of only three relative equilibria,  $\xi_j$ ,  $j = 0, 1, 2$ , corresponding to the  $\Gamma$ -orbit of the equilibria  $(R, P)$ ,  $(R, S)$  and  $(R, R)$ , respectively, i.e.,

$$\begin{aligned} \xi_0 &\equiv \Gamma(R, P) = \{(R, P), (S, R), (P, S)\} \\ \xi_1 &\equiv \Gamma(R, S) = \{(R, S), (S, P), (P, R)\} \\ \xi_2 &\equiv \Gamma(R, R) = \{(R, R), (S, S), (P, P)\}. \end{aligned}$$

From the perspective of the game, agent  $Y$  is the winner along  $\xi_0$ , agent  $X$  is the winner along  $\xi_1$ , whereas a tie takes place along  $\xi_2$ .

Now the quotient heteroclinic cycles  $C_i$ ,  $i = 0, 1, 2, 3, 4$ , have the following architecture:  $C_0$  involving relative equilibria  $\xi_0$  and  $\xi_1$ ,  $C_1$  involving relative equilibria  $\xi_1$  and  $\xi_2$ ,  $C_2$  involving relative equilibria  $\xi_0$  and  $\xi_2$ ,  $C_3$  sequentially

Connection	Representative	2-dim space $P_{kj}$	3-dim vector space $Q_{kj}$
$[\xi_0 \rightarrow \xi_1]$	$[(R, P) \rightarrow (S, P)]$	$\{(x_1, x_2, 0; 0, 0, 1)\}$	$\{(x_1, x_2, 0; 0, 0, y_3)\}$
$[\xi_1 \rightarrow \xi_0]$	$[(S, P) \rightarrow (S, R)]$	$\{(0, 1, 0; y_1, 0, y_3)\}$	$\{(0, x_2, 0; y_1, 0, y_3)\}$
$[\xi_1 \rightarrow \xi_2]$	$[(R, S) \rightarrow (R, R)]$	$\{(1, 0, 0; y_1, y_2, 0)\}$	$\{(x_1, 0, 0; y_1, y_2, 0)\}$
$[\xi_2 \rightarrow \xi_1]$	$[(R, R) \rightarrow (P, R)]$	$\{(x_1, 0, x_3; 1, 0, 0)\}$	$\{(x_1, 0, x_3; y_1, 0, 0)\}$
$[\xi_0 \rightarrow \xi_2]$	$[(S, R) \rightarrow (R, R)]$	$\{(x_1, x_2, 0; 1, 0, 0)\}$	$\{(x_1, x_2, 0; y_1, 0, 0)\}$
$[\xi_2 \rightarrow \xi_0]$	$[(R, R) \rightarrow (R, P)]$	$\{(1, 0, 0; y_1, 0, y_3)\}$	$\{(x_1, 0, 0; y_1, 0, y_3)\}$

Table 1: Flow-invariant subspaces and representatives for connections in the quotient network.

involving relative equilibria  $\xi_0$ ,  $\xi_1$  and  $\xi_2$ ,  $C_4$  sequentially involving relative equilibria  $\xi_0$ ,  $\xi_2$  and  $\xi_1$ . Figure 1 illustrates the connections in the cycles and in the network as a whole.

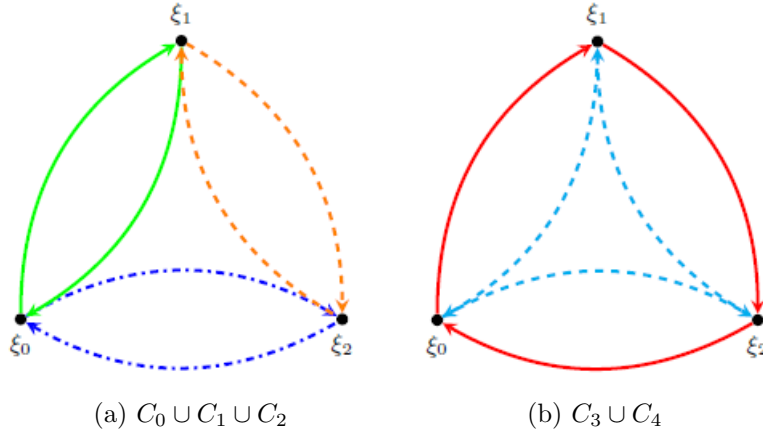


Figure 1: Cycles in the quotient heteroclinic network. Each style identifies a quotient heteroclinic cycle: (a)  $C_0$  is represented by a green solid line,  $C_1$  by an orange dashed line, and  $C_2$  by a blue dash-dot line; (b)  $C_3$  is represented by a red solid line, and  $C_4$  by a cyan dashed line.

Let  $P_{kj}$  be a two-dimensional flow-invariant subspace as above such that  $[\xi_k \rightarrow \xi_j] \subset P_{kj}$ ,  $j \neq k = 0, 1, 2$ . We easily find a three-dimensional vector subspace  $Q_{kj}$  invariant under the flow such that  $P_{kj} \subset Q_{kj}$  and  $\xi_k$  is a saddle and  $\xi_j$  is a sink in  $Q_{kj}$ . Representatives of the connections in the quotient network and respective flow-invariant subspaces that contain them are listed in Table 1. Set  $L_j = Q_{kj} \cap Q_{jl}$ , for  $j \neq k, l = 0, 1, 2$ . Evidently,  $\xi_j \in L_j$ .

We are interested in stability properties of the quotient cycles  $C_i$ ,  $i = 0, 1, 2, 3, 4$ . Generically, stability of robust heteroclinic cycle depends on the

Equilibria	$e_1$	$e_2$	$e_3$	$e_4$	$e_5$	$e_6$
$\xi_0$	$1 + \frac{\varepsilon_x}{3}$	2	$1 + \varepsilon_x$	$-1 + \varepsilon_y$	-2	$-1 + \frac{\varepsilon_y}{3}$
$\xi_1$	$-1 + \frac{\varepsilon_x}{3}$	$-1 + \varepsilon_x$	-2	$1 + \varepsilon_y$	$1 + \frac{\varepsilon_y}{3}$	2
$\xi_2$	$-\frac{2\varepsilon_x}{3}$	$-1 - \varepsilon_x$	$1 - \varepsilon_y$	$-\frac{2\varepsilon_y}{3}$	$-1 - \varepsilon_y$	$1 - \varepsilon_x$

Table 2: Eigenvalues with respect to local coordinates at each equilibrium of the network.

relative sizes of certain eigenvalues of the Jacobian matrix associated with the flow linearized about each equilibrium along the cycle. The eigenvalues of the Jacobian matrix at  $\xi_j$  ( $j = 0, 1, 2$ ) are all real and distinct. Table 2 presents the eigenvalues and eigenvectors for the three relative equilibria  $\xi_0, \xi_1, \xi_2$ , as representatives the equilibria  $(R, P)$ ,  $(R, S)$ ,  $(R, R)$ , respectively. According to the classification of eigenvalues with respect to a heteroclinic cycle, for each relative equilibrium  $\xi_j$ , both radial eigenspace  $L_j$  and transverse eigenspace  $V_j(t) = (Q_{kj} + Q_{jl})^\perp$  are two-dimensional and both contracting eigenspace  $V_j(c) = Q_{kj} \ominus L_j$  and expanding eigenspace  $V_j(e) = Q_{jl} \ominus L_j$  are one-dimensional. In other words, the sets of radial and transverse eigenvalues have each two elements and the sets of contracting and expanding eigenvalues have each one element. Therefore, at each relative equilibrium  $\xi_j$ , we denote by  $-r_{jX}$  and  $-r_{jY}$  the radial eigenvalues,  $-c_{jk}$  the eigenvalue in the stable direction through the connection  $[\xi_k \rightarrow \xi_j]$ ,  $e_{jl}$  the eigenvalue in the unstable direction through the connection  $[\xi_j \rightarrow \xi_l]$ , for some positive constants  $r_{jX}$ ,  $r_{jY}$ ,  $c_{jk}$ ,  $e_{jl}$ ,  $j \neq k, l = 0, 1, 2$ . In particular, the transverse eigenvalues have opposite signs and are of the form  $-c_{jm}$  and  $e_{js}$ ,  $j \neq m, s = 0, 1, 2$ .

## 4 Transitions near the RSP cycles

In this section, we depict the construction of the maps required for modelling the transitions along the cycles in the heteroclinic network. These provide a good approximate description of the dynamics near each heteroclinic cycle and constitute the basis for the study of stability that follows in Section 5.

Specifically, we construct Poincaré maps used to model the behaviour of trajectories near either cycle. The Poincaré maps are obtained via the composition of local and global maps. The local maps approximate the flow in a neighbourhood of an equilibrium while the global maps approximate the flow along a heteroclinic connection between two consecutive equilibria. Following standard techniques as in [15], we define coordinates and cross

sections about the equilibria  $\xi_j$ ,  $j = 0, 1, 2$ , and determine the local and global maps required to the suitable set-up for the dynamics. In particular, these maps are represented in local coordinates near equilibria. When convenient, we use logarithmic coordinates which are especially useful to describe the dynamics via transition matrices.

## 4.1 Poincaré maps

In a neighbourhood of each equilibrium  $\xi_j$  ( $j = 0, 1, 2$ ), we choose local coordinates derived from the translation of  $\xi_j$  into the origin and such that coordinate axes are aligned with the eigenvectors of the Jacobian matrix at  $\xi_j$ . Denote by  $(u_1, u_2, v, w, z_1, z_2)$  local coordinates about  $\xi_j$  in the basis whose two radial eigenvectors come first, followed by one contracting, one expanding and two transverse eigenvectors.

We introduce a five-dimensional cross section transverse to the flow for each of the four connections near the equilibria. For  $\xi_j$  and  $j \neq k = 0, 1, 2$ , we define  $H_j^{in,k}$  as the cross section near  $\xi_j$  at  $[\xi_k \rightarrow \xi_j]$  and  $H_j^{out,k}$  as the cross section near  $\xi_j$  at  $[\xi_j \rightarrow \xi_k]$ , where

$$\begin{aligned} H_j^{in,k} &= \{(u_1, u_2, v, w, z_1, z_2) : |u_1|, |u_2| < 1, 0 \leq w, z_1, z_2 < 1, v = 1\} \\ H_j^{out,k} &= \{(u_1, u_2, v, w, z_1, z_2) : |u_1|, |u_2| < 1, 0 \leq v, z_1, z_2 < 1, w = 1\}. \end{aligned}$$

In local coordinates near  $\xi_j$ , the linearized flow is given by

$$\begin{aligned} \dot{u}_1 &= -r_j X u_1 \\ \dot{u}_2 &= -r_j Y u_2 \\ \dot{v} &= -c_{jk} v \\ \dot{w} &= e_{jl} w \\ \dot{z}_1 &= -c_{jm} z_1 \\ \dot{z}_2 &= e_{js} z_2, \end{aligned} \tag{2}$$

for  $j \neq k, l, m, s = 0, 1, 2$ , with  $k \neq m$ ,  $l \neq s$ . The type of the eigenvalues depends on the heteroclinic cycle under consideration. Table 3 provides all transverse eigenvalues.

We define a local map (also called first hit map)  $\phi_{kjl} : H_j^{in,k} \rightarrow H_j^{out,l}$  in a neighbourhood of  $\xi_j$  and a global map (also called connecting diffeomorphisms)  $\psi_{jl} : H_j^{out,l} \rightarrow H_l^{in,j}$  along the connection  $[\xi_j \rightarrow \xi_l] \subset Q_{jl}$ . Denote the composition of the local  $\phi_{kjl}$  and the global  $\psi_{jl}$  maps by  $\tilde{g}_{kjl} = \psi_{jl} \circ \phi_{kjl} : H_j^{in,k} \rightarrow H_l^{in,j}$ . Then a Poincaré map  $\tilde{\pi}_j : H_j^{in,l} \rightarrow H_j^{in,l}$  with respect to a 2-cycle as  $[\xi_l \rightarrow \xi_j \rightarrow \xi_l]$  is the composition  $\tilde{\pi}_j = \tilde{g}_{jll} \circ \tilde{g}_{ljl}$ , for  $j \neq l = 0, 1, 2$ ,

Equilibrium	$C_0$ -cycle		$C_1$ -cycle		$C_2$ -cycle		$C_3$ -cycle		$C_4$ -cycle	
$\xi_0$	$-c_{02}$	$e_{02}$			$-c_{01}$	$e_{01}$	$-c_{01}$	$e_{02}$	$-c_{02}$	$e_{01}$
$\xi_1$	$-c_{12}$	$e_{12}$	$-c_{10}$	$e_{10}$			$-c_{12}$	$e_{10}$	$-c_{10}$	$e_{12}$
$\xi_2$			$-c_{20}$	$e_{20}$	$-c_{21}$	$e_{21}$	$-c_{20}$	$e_{21}$	$-c_{21}$	$e_{20}$

Table 3: Transverse eigenvalues at the equilibria of each cycle in the RSP network.

whereas, with respect to a 3-cycle as  $[\xi_k \rightarrow \xi_j \rightarrow \xi_l \rightarrow \xi_k]$ , is the composition  $\tilde{\pi}_j = \tilde{g}_{lkj} \circ \tilde{g}_{jlk} \circ \tilde{g}_{kjl}$ , for  $j \neq k, l = 0, 1, 2$ . In other words,  $\tilde{\pi}_j$  characterizes the behaviour of trajectories around the cycle from  $\xi_j$  to itself. A heteroclinic cycle is henceforth identified with a collection of Poincaré maps, one for each connection. We write  $\tilde{\pi}_{j,i}$  for a Poincaré map associated with the  $C_i$ -cycle,  $i = 0, 1, 2, 3, 4$ .

We integrate (2) and replace the time variable by the time of flight of the trajectory from  $H_j^{in,k}$  to  $H_j^{out,l}$  to obtain

$$\phi_{kjl}(u_1, u_2, 1, w, z_1, z_2) = \left( u_1 w^{\frac{r_{jX}}{e_{jl}}}, u_2 w^{\frac{r_{jY}}{e_{jl}}}, w^{\frac{c_{jk}}{e_{jl}}}, 1, z_1 w^{\frac{c_{jm}}{e_{jl}}}, z_2 w^{-\frac{e_{js}}{e_{jl}}} \right),$$

for  $0 \leq z_2 < w^{\frac{e_{j,s}}{e_{j,l}}}$ <sup>3</sup>. Given any initial condition  $(u_1^0, u_2^0, 1, 0, 0, 0) \in Q_{jl} \cap H_l^{in,j}$ , with  $u_1^0, u_2^0 \neq 0$ , sufficiently close to  $[\xi_j \rightarrow \xi_l] \cap H_l^{in,j}$ , the local map  $\phi_{kjl}$  turns out to depend only on  $w, z_1$  and  $z_2$  components. Consequently, the computation of  $\tilde{g}_{kjl}$  as well as the Poincaré maps  $\tilde{\pi}_j$  simply require  $(w, z_1, z_2)$  coordinates.

We construct the global map  $\psi_{jl}$  taking into account the invariance of the three-dimensional subspaces  $Q_{jl}$  whence

$$\psi_{jl}(Q_{jl} \cap H_j^{out,l}) \subset Q_{jl} \cap H_l^{in,j}, \quad (3)$$

with

$$Q_{jl} = \begin{cases} \{(u_1, u_2, 0, w, 0, 0) : u_1, u_2, w \in \mathbb{R}\} & \text{near } \xi_j \\ \{(u_1, u_2, v, 0, 0, 0) : u_1, u_2, v \in \mathbb{R}\} & \text{near } \xi_l. \end{cases}$$

Write  $\psi_{jl}$  in components  $\psi_{jl} = (\psi_{jl}^{u_1}, \psi_{jl}^{u_2}, \psi_{jl}^v, \psi_{jl}^w, \psi_{jl}^{z_1}, \psi_{jl}^{z_2})$ . Hence, by virtue

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<sup>3</sup>Actually, the domain of definition of the map  $\phi_{kjl}$  is constrained by the inequality  $(1 - \varepsilon) w^{\frac{e_{j,s}}{e_{j,l}}} > z_2 \geq 0$ , for a small positive constant  $\varepsilon$ , which in turn is irrelevant for the study of stability of a heteroclinic cycle (see [15], [23], [24]).

of (3), it follows that

$$\psi_{jl}(u_1, u_2, 0, w, 0, 0) \equiv \begin{pmatrix} \psi_{jl}^{u_1}(u_1, u_2, 0, w, 0, 0) \\ \psi_{jl}^{u_2}(u_1, u_2, 0, w, 0, 0) \\ \psi_{jl}^v(u_1, u_2, 0, w, 0, 0) \\ \psi_{jl}^w(u_1, u_2, 0, w, 0, 0) \\ \psi_{jl}^{z_1}(u_1, u_2, 0, w, 0, 0) \\ \psi_{jl}^{z_2}(u_1, u_2, 0, w, 0, 0) \end{pmatrix} = \begin{pmatrix} \psi_{jl}^{u_1}(u_1, u_2, 0, w, 0, 0) \\ \psi_{jl}^{u_2}(u_1, u_2, 0, w, 0, 0) \\ \psi_{jl}^v(u_1, u_2, 0, w, 0, 0) \\ 0 \\ 0 \\ 0 \end{pmatrix}.$$

Retaining just the terms of lowest order, we find

$$\begin{aligned} \psi_{jl}^w(u_1, u_2, v, w, z_1, z_2) &= \alpha_{jl,1}v + \alpha_{jl,2}z_1 + \alpha_{jl,3}z_2 \\ \psi_{jl}^{z_1}(u_1, u_2, v, w, z_1, z_2) &= \alpha_{jl,4}v + \alpha_{jl,5}z_1 + \alpha_{jl,6}z_2 \\ \psi_{jl}^{z_2}(u_1, u_2, v, w, z_1, z_2) &= \alpha_{jl,7}v + \alpha_{jl,8}z_1 + \alpha_{jl,9}z_2, \end{aligned}$$

for some  $\alpha_{jl,q} \in \mathbb{R}$ ,  $q = 1, \dots, 9$ . We have already shown that radial components play no role for the transition maps so  $\psi_{jl}^{u_1}$  and  $\psi_{jl}^{u_2}$  are here insignificant. The specific form of the global maps depend on the connection. We distinguish three cases:

- (i) If the connection is of type  $[\xi_j \rightarrow \xi_l]$ , with  $j \neq l \pmod 2$ , then  $\alpha_{jl,1} = \alpha_{jl,3} = \alpha_{jl,4} = \alpha_{jl,5} = \alpha_{jl,8} = \alpha_{jl,9} = 0$  and  $\alpha_{jl,2}, \alpha_{jl,6}, \alpha_{jl,7} \neq 0$ , and

$$\begin{aligned} \psi_{jl}^w(u_1, u_2, v, w, z_1, z_2) &= \alpha_{jl,2}z_1 \\ \psi_{jl}^{z_1}(u_1, u_2, v, w, z_1, z_2) &= \alpha_{jl,6}z_2 \\ \psi_{jl}^{z_2}(u_1, u_2, v, w, z_1, z_2) &= \alpha_{jl,7}v; \end{aligned}$$

- (ii) If the connection is of type  $[\xi_j \rightarrow \xi_2]$ , with  $j \pmod 2$ , then  $\alpha_{j2,1} = \alpha_{j2,2} = \alpha_{j2,4} = \alpha_{j2,6} = \alpha_{j2,8} = \alpha_{j2,9} = 0$  and  $\alpha_{j2,3}, \alpha_{j2,5}, \alpha_{j2,7} \neq 0$ , and

$$\begin{aligned} \psi_{j2}^w(u_1, u_2, v, w, z_1, z_2) &= \alpha_{j2,3}z_2 \\ \psi_{j2}^{z_1}(u_1, u_2, v, w, z_1, z_2) &= \alpha_{j2,5}z_1 \\ \psi_{j2}^{z_2}(u_1, u_2, v, w, z_1, z_2) &= \alpha_{j2,7}v; \end{aligned}$$

- (iii) If the connection is of type  $[\xi_2 \rightarrow \xi_l]$ , with  $l \pmod 2$ , then  $\alpha_{2l,1} = \alpha_{2l,3} = \alpha_{2l,5} = \alpha_{2l,6} = \alpha_{2l,7} = \alpha_{2l,8} = 0$  and  $\alpha_{2l,2}, \alpha_{2l,4}, \alpha_{2l,9} \neq 0$ , and

$$\begin{aligned} \psi_{2l}^w(u_1, u_2, v, w, z_1, z_2) &= \alpha_{2l,2}z_1 \\ \psi_{2l}^{z_1}(u_1, u_2, v, w, z_1, z_2) &= \alpha_{2l,4}v \\ \psi_{2l}^{z_2}(u_1, u_2, v, w, z_1, z_2) &= \alpha_{2l,9}z_2. \end{aligned}$$

The flow invariance of the state space  $\Delta$  contained in the non-negative orthant of  $\mathbb{R}^6$  ensures that the nonzero constants  $\alpha_{jl,q}$  are all positive and  $O(1)$ .

Notice that  $\mathbb{R}^6 = Q_{jl} \oplus Q_{jl}^\perp$  where both  $Q_{jl}$  and  $Q_{jl}^\perp$  are flow-invariant subspaces. We look at  $g_{kjl} : \mathbb{R}^3 \rightarrow \mathbb{R}^3$  and  $\pi_j : \mathbb{R}^3 \rightarrow \mathbb{R}^3$  as the restrictions of the maps  $\tilde{g}_{kjl}$  and  $\tilde{\pi}_j$ , into the  $(w, z_1, z_2)$ -subspace corresponding exactly to  $Q_{jl}^\perp = \{(0, 0, 0, w, z_1, z_2) : w, z_1, z_2 \in \mathbb{R}\} \cong \mathbb{R}^3$ . Furthermore, as observed by [6], constants arising from local and global maps are irrelevant for the stability indices so we normalize them to unity. Thus, we differentially write down

$$\begin{aligned} \text{(i)} \quad g_{kjl}(w, z_1, z_2) &= \left( z_1 w^{\frac{c_{jm}}{e_{jl}}}, z_2 w^{-\frac{e_{js}}{e_{jl}}}, w^{\frac{c_{jk}}{e_{jl}}} \right); \\ \text{(ii)} \quad g_{kjl}(w, z_1, z_2) &= \left( z_2 w^{-\frac{e_{js}}{e_{jl}}}, z_1 w^{\frac{c_{jm}}{e_{jl}}}, w^{\frac{c_{jk}}{e_{jl}}} \right); \\ \text{(iii)} \quad g_{kjl}(w, z_1, z_2) &= \left( z_1 w^{\frac{c_{jm}}{e_{jl}}}, w^{\frac{c_{jk}}{e_{jl}}}, z_2 w^{-\frac{e_{js}}{e_{jl}}} \right), \end{aligned}$$

for  $0 \leq z_2 < w^{\frac{e_{js}}{e_{jl}}}$ . All maps  $g_{kjl}$  for  $j \neq k, l = 0, 1, 2$  are set out in Appendix A based on the eigenvalues of  $\xi_j$  in Table 2.

We provide a detailed description of the study of the cycle  $C_0 = [\xi_0 \rightarrow \xi_1 \rightarrow \xi_0]$ . From here, the information concerning the remaining heteroclinic cycles in the Appendices A–C suffices to complete the analysis of the properties of all cycles. The dynamics of trajectories close to the  $C_0$ -cycle is modeled by two reduced Poincaré maps  $\pi_{0,0} = g_{010} \circ g_{101}$  and  $\pi_{1,0} = g_{101} \circ g_{010}$  with

$$\begin{aligned} g_{101}(w, z_1, z_2) &= \left( z_1 w^{\frac{c_{02}}{e_{01}}}, z_2 w^{-\frac{e_{02}}{e_{01}}}, w^{\frac{c_{01}}{e_{01}}} \right), \quad 0 \leq z_2 < w^{\frac{e_{02}}{e_{01}}} \\ g_{010}(w, z_1, z_2) &= \left( z_1 w^{\frac{c_{12}}{e_{10}}}, z_2 w^{-\frac{e_{12}}{e_{10}}}, w^{\frac{c_{10}}{e_{10}}} \right), \quad 0 \leq z_2 < w^{\frac{e_{12}}{e_{10}}} \end{aligned} \quad (4)$$

and

$$\begin{aligned} \pi_{0,0}(w, z_1, z_2) &= \left( z_2 z_1^{\frac{c_{12}}{e_{10}}} w^{-\frac{e_{02}}{e_{01}} + \frac{c_{02}}{e_{01}} \frac{c_{12}}{e_{20}}}, z_1^{\frac{e_{12}}{e_{10}}} w^{\frac{c_{01}}{e_{01}} - \frac{c_{02}}{e_{01}} \frac{e_{12}}{e_{10}}}, z_1^{\frac{c_{10}}{e_{10}}} w^{\frac{c_{02}}{e_{01}} \frac{c_{10}}{e_{10}}} \right), \\ &\quad 0 \leq z_2 < w^{\frac{e_{02}}{e_{01}}} \wedge z_1 > w^{\frac{c_{01}}{e_{01}} \frac{e_{10}}{e_{12}} - \frac{c_{02}}{e_{01}}}, \\ \pi_{1,0}(w, z_1, z_2) &= \left( z_2 z_1^{\frac{c_{02}}{e_{01}}} w^{-\frac{e_{12}}{e_{10}} + \frac{c_{12}}{e_{10}} \frac{c_{02}}{e_{01}}}, z_1^{\frac{e_{02}}{e_{01}}} w^{\frac{c_{10}}{e_{10}} - \frac{c_{12}}{e_{10}} \frac{e_{02}}{e_{01}}}, z_1^{\frac{c_{01}}{e_{01}}} w^{\frac{c_{12}}{e_{10}} \frac{c_{01}}{e_{01}}} \right), \\ &\quad 0 \leq z_2 < w^{\frac{e_{12}}{e_{10}}} \wedge z_1 > w^{\frac{c_{10}}{e_{10}} \frac{e_{01}}{e_{02}} - \frac{c_{12}}{e_{10}}}. \end{aligned} \quad (5)$$

The origin  $(w, z_1, z_2) = \mathbf{0}$  is a fixed point of the maps  $\pi_{j,i}$  and corresponds to the heteroclinic cycle  $C_i$  in the flow. Hence, the stability of a heteroclinic cycle follows from the stability of the fixed point at the origin of a collection of reduced Poincaré maps associated with the cycle. On other hand, attraction properties of heteroclinic cycles (an invariant set, in general) are defined in terms of basins of attraction. Following Podvigina and Ashwin [24], for  $\delta > 0$ , we denote by  $\mathcal{B}_\delta^{\pi_j}$  the  $\delta$ -local basin of attraction of  $(w, z_1, z_2) = \mathbf{0}$  in  $\mathbb{R}^3$  for a reduced Poincaré map  $\pi_j$  associated with a heteroclinic cycle. The set  $\mathcal{B}_\delta^{\pi_j}$  represents the points that remains in a  $\delta$ - neighbourhood of the cycle for any iteration of the return map  $\pi_j$  and partial turns through sequences of connections from  $\xi_j$  after any number of full returns. In particular, for the map  $\pi_j = g_{jlj} \circ g_{ljl}$  with respect to a 2-cycle  $[\xi_l \rightarrow \xi_j \rightarrow \xi_l]$ , we have

$$\mathcal{B}_\delta^{\pi_j} = \left\{ (w, z_1, z_2) \in \mathbb{R}^3 : \begin{aligned} &\|\pi_j^n(w, z_1, z_2)\| < \delta, \\ &\|g_{ljl} \circ \pi_j^n(w, z_1, z_2)\| < \delta, \quad \forall n \in \mathbb{N}_0 \end{aligned} \right\} \quad (6)$$

and for the map  $\pi_j = g_{lkj} \circ g_{jlk} \circ g_{kjl}$  with respect to a 3-cycle  $[\xi_k \rightarrow \xi_j \rightarrow \xi_l \rightarrow \xi_k]$ ,

$$\mathcal{B}_\delta^{\pi_j} = \left\{ (w, z_1, z_2) \in \mathbb{R}^3 : \begin{aligned} &\|\pi_j^n(w, z_1, z_2)\| < \delta, \\ &\|g_{kjl} \circ \pi_j^n(w, z_1, z_2)\| < \delta, \\ &\|g_{jlk} \circ g_{kjl} \circ \pi_j^n(w, z_1, z_2)\| < \delta, \quad \forall n \in \mathbb{N}_0 \end{aligned} \right\}. \quad (7)$$

## 4.2 Transition matrices

Consider the change of coordinates

$$\boldsymbol{\eta} \equiv (\eta_1, \eta_2, \eta_3) = (\ln(v), \ln(z_1), \ln(z_2)).$$

We denote by  $M_{kjl}$  ( $j \neq l = 0, 1, 2$ ) the transition matrix of the map  $g_{kjl}$  in the new coordinates where  $g_{kjl}(\boldsymbol{\eta}) = M_{kjl}\boldsymbol{\eta}$  such that

$$(i) \quad M_{kjl} = \begin{bmatrix} \frac{c_{jm}}{e_{jl}} & 1 & 0 \\ -\frac{e_{js}}{e_{jl}} & 0 & 1 \\ \frac{c_{jk}}{e_{jl}} & 0 & 0 \end{bmatrix} \quad (ii) \quad M_{kjl} = \begin{bmatrix} -\frac{e_{js}}{e_{jl}} & 0 & 1 \\ \frac{c_{jm}}{e_{jl}} & 1 & 0 \\ \frac{c_{jk}}{e_{jl}} & 0 & 0 \end{bmatrix} \quad (iii) \quad M_{kjl} = \begin{bmatrix} \frac{c_{jm}}{e_{jl}} & 1 & 0 \\ \frac{c_{jk}}{e_{jl}} & 0 & 0 \\ -\frac{e_{js}}{e_{jl}} & 0 & 1 \end{bmatrix}.$$

As a consequence, the transition matrix of the composition of maps either  $\pi_j = g_{jlj} \circ g_{ljl}$  (2-cycle) or  $\pi_j = g_{lkj} \circ g_{jlk} \circ g_{kjl}$  (3-cycle) is the product  $M^{(j)} = M_{jlj}M_{ljl}$  and  $M^{(j)} = M_{lkj}M_{jlk}M_{kjl}$ , respectively. We write  $M_i^{(j)}$



for the transition matrix associated with the  $C_i$ -cycle,  $i = 0, 1, 2, 3, 4$ . For example, taking into account the eigenvalues of  $\xi_j$ ,  $j = 0, 1, 2$ , in Table 2, transition matrices of the maps (4) and (5) associated with the  $C_0$ -cycle are given by

$$\begin{aligned} M_{101} &= \begin{bmatrix} \frac{c_{02}}{e_{01}} & 1 & 0 \\ -\frac{e_{02}}{e_{01}} & 0 & 1 \\ \frac{c_{01}}{e_{01}} & 0 & 0 \end{bmatrix} = \begin{bmatrix} \frac{1-\varepsilon_y}{2} & 1 & 0 \\ -\frac{1+\varepsilon_x}{2} & 0 & 1 \\ 1 & 0 & 0 \end{bmatrix} \\ M_{010} &= \begin{bmatrix} \frac{c_{12}}{e_{10}} & 1 & 0 \\ -\frac{e_{12}}{e_{01}} & 0 & 1 \\ \frac{c_{10}}{e_{10}} & 0 & 0 \end{bmatrix} = \begin{bmatrix} \frac{1-\varepsilon_x}{2} & 1 & 0 \\ -\frac{1+\varepsilon_y}{2} & 0 & 1 \\ 1 & 0 & 0 \end{bmatrix}, \end{aligned} \quad (8)$$

and

$$\begin{aligned} M_0^{(0)} &= M_{010}M_{101} = \begin{bmatrix} \frac{-1-3\varepsilon_x-\varepsilon_y+\varepsilon_x\varepsilon_y}{4} & \frac{1-\varepsilon_x}{2} & 1 \\ \frac{3+\varepsilon_y^2}{4} & -\frac{1+\varepsilon_y}{2} & 0 \\ \frac{1-\varepsilon_y}{2} & 1 & 0 \end{bmatrix} \\ M_0^{(1)} &= M_{101}M_{010} = \begin{bmatrix} \frac{-1-\varepsilon_x-3\varepsilon_y+\varepsilon_x\varepsilon_y}{4} & \frac{1-\varepsilon_y}{2} & 1 \\ \frac{3+\varepsilon_x^2}{4} & -\frac{1+\varepsilon_x}{2} & 0 \\ \frac{1-\varepsilon_x}{2} & 1 & 0 \end{bmatrix}. \end{aligned} \quad (9)$$

In a similar way, we determine the transition matrices for the remaining cycles  $C_i$ ,  $i = 1, 2, 3, 4$ , which are listed in Appendix C.

The fixed point  $(w, z_1, z_2) = \mathbf{0}$  of the reduced Poincaré maps  $\pi_{j,i}$  associated with the heteroclinic cycle  $C_i$  becomes the fixed point  $\boldsymbol{\eta} = -\infty$  of the linear maps  $M_i^{(j)}$ . Accordingly, the study of the stability of the point  $(w, z_1, z_2) = \mathbf{0}$  assumes that  $w, z_1, z_2$  are asymptotically small, which corresponds to asymptotically large negative  $\eta_1, \eta_2, \eta_3$ . Furthermore, the  $\delta$ -local basin of attraction  $\mathcal{B}_\delta^{\pi_j}$  in logarithmic coordinates consists in the points

$\boldsymbol{\eta} \in \mathbb{R}_-^3$  such that either

$$\begin{aligned} (M^{(j)})^n \boldsymbol{\eta} &\xrightarrow{n \rightarrow \infty} -\infty \\ M_{jl} (M^{(j)})^n \boldsymbol{\eta} &\xrightarrow{n \rightarrow \infty} -\infty \end{aligned} \quad (10)$$

or

$$\begin{aligned} (M^{(j)})^n \boldsymbol{\eta} &\xrightarrow{n \rightarrow \infty} -\infty \\ M_{jl} (M^{(j)})^n \boldsymbol{\eta} &\xrightarrow{n \rightarrow \infty} -\infty \\ M_{lk} M_{jl} (M^{(j)})^n \boldsymbol{\eta} &\xrightarrow{n \rightarrow \infty} -\infty \end{aligned}$$

depending on whether the cycle connects two (6) or three (7) equilibria, respectively. Denote

$$U^{-\infty}(M^{(j)}) = \left\{ \mathbf{y} = (y_1, y_2, y_3) : \mathbf{y} \in \mathbb{R}_-^3, \lim_{n \rightarrow +\infty} (M^{(j)})^n \mathbf{y} = -\infty \right\}.$$

From here, a necessary condition for  $(w, z_1, z_2) \in \mathcal{B}_\delta^{\pi_j}$  is that  $\boldsymbol{\eta} \in U^{-\infty}(M^{(j)})$ . Podvigina [23] establishes a criterion for positivity of the measure of the set  $U^{-\infty}(M)$  for an arbitrary  $N \times N$  ( $N \in \mathbb{N}$ ) real matrix  $M$ , in terms of  $\lambda_{\max}$  and  $\mathbf{w}^{\max} = (w_1^{\max}, \dots, w_N^{\max})$ , the maximum, in absolute value, eigenvalue of  $M$  and its associated eigenvector. More generally, consider a linear map  $\mathcal{M} : \mathbb{R}^N \rightarrow \mathbb{R}^N$  whose the image of each point  $\boldsymbol{\zeta} \in \mathbb{R}^N$  in the domain of  $\mathcal{M}$  is  $M\boldsymbol{\zeta} + C$ , for some  $C \in \mathbb{R}^N$ .

**Lemma 4.1** (Lemma 5 in [23]). *Let  $\lambda_{\max}$  be the largest in absolute value significant eigenvalue of the matrix  $M$  in  $\mathcal{M}\boldsymbol{\zeta} = M\boldsymbol{\zeta} + C$  and  $\mathbf{w}^{\max}$  be the associated eigenvector. Suppose  $\lambda_{\max} \neq 1$ . The measure  $\ell(U^{-\infty}(\mathcal{M}))$  is positive, if and only if the three following conditions are satisfied:*

- (i)  $\lambda_{\max}$  is real;
- (ii)  $\lambda_{\max} > 1$ ;
- (iii)  $w_l^{\max} w_q^{\max} > 1$  for all  $l$  and  $q$ ,  $1 \leq l, q \leq N$ .

In particular, we have  $C = \mathbf{0}$  because we omit all constants arising from local and global maps. Otherwise, we can always deal with matrix  $M$  rather than  $\mathcal{M}$  due to [11, Theorem 3.4] by showing that the sets  $U^{-\infty}(M)$  and  $U^{-\infty}(\mathcal{M})$  coincide under certain circumstances which are controlled by initial conditions.

We combine transition matrices and stability index to discuss the stability and attractiveness of the RSP cycles.

## 5 Stability of the RSP cycles

In what follows, we study the stability of the heteroclinic cycles  $C_i$ ,  $i = 0, 1, 2, 3, 4$ , by applying the stability index of Podvigina and Ashwin [24]. The set of stability indices of individual connections describes the local attraction properties of a heteroclinic cycle.

We use the results of [11] which apply to the calculation of local stability indices for generic heteroclinic cycles with one-dimensional connections and such that the dynamics along each connection is close to the identity. Notice that all  $C_i$ -cycles,  $i = 0, 1, 2, 3, 4$ , satisfy these assumptions. To calculate stability indices, we introduce the function  $F^{\text{index}} : \mathbb{R}^3 \setminus \{\boldsymbol{\alpha} = (\alpha_1, \alpha_2, \alpha_3) \in \mathbb{R}^3 : \alpha_1 + \alpha_2 + \alpha_3 = 0\} \rightarrow [-\infty, \infty]$ , where  $F^{\text{index}}(\boldsymbol{\alpha})$  represents the local stability index for a return map  $\mathbb{R}^3 \rightarrow \mathbb{R}^3$  whose  $\delta$ -local basin of attraction of  $\mathbf{0}$  is of the form  $\mathcal{B}_\delta = \{\mathbf{x} = (x_1, x_2, x_3) \in \mathbb{R}^3 : |x_1^{\alpha_1} x_2^{\alpha_2} x_3^{\alpha_3}| < \delta\}$ . Then,

$$F^{\text{index}}(\boldsymbol{\alpha}) = F^+(\boldsymbol{\alpha}) - F^-(\boldsymbol{\alpha})$$

with

$$F^+(\boldsymbol{\alpha}) = \begin{cases} \infty, & \alpha_1 \geq 0, \alpha_2 \geq 0, \alpha_3 \geq 0 \\ 0, & \alpha_1 \leq 0, \alpha_2 \leq 0, \alpha_3 \leq 0 \\ -\frac{\alpha_2 + \alpha_3}{\alpha_1} - 1, & \alpha_1 < 0, \alpha_2 \geq 0, \alpha_3 \geq 0, \frac{\alpha_2 + \alpha_3}{\alpha_1} < -1 \\ 0, & \alpha_1 > 0, \alpha_2 \leq 0, \alpha_3 \leq 0, \frac{\alpha_2 + \alpha_3}{\alpha_1} < -1 \\ \min\left\{-\frac{\alpha_1 + \alpha_3}{\alpha_2}, -\frac{\alpha_1 + \alpha_2}{\alpha_3}\right\} - 1, & \alpha_1 > 0, \alpha_2 < 0, \alpha_3 < 0, \frac{\alpha_2 + \alpha_3}{\alpha_1} > -1 \\ 0, & \alpha_1 < 0, \alpha_2 > 0, \alpha_3 > 0, \frac{\alpha_2 + \alpha_3}{\alpha_1} > -1 \\ -\frac{\alpha_1 + \alpha_3}{\alpha_2} - 1, & \alpha_1 \geq 0, \alpha_2 < 0, \alpha_3 \geq 0, \frac{\alpha_1 + \alpha_3}{\alpha_2} < -1 \\ 0, & \alpha_1 \leq 0, \alpha_2 > 0, \alpha_3 \leq 0, \frac{\alpha_1 + \alpha_3}{\alpha_2} < -1 \\ \min\left\{-\frac{\alpha_1 + \alpha_3}{\alpha_2}, -\frac{\alpha_1 + \alpha_2}{\alpha_3}\right\} - 1, & \alpha_1 < 0, \alpha_2 > 0, \alpha_3 < 0, \frac{\alpha_1 + \alpha_3}{\alpha_2} > -1 \\ 0, & \alpha_1 < 0, \alpha_2 < 0, \alpha_3 > 0, \frac{\alpha_1 + \alpha_3}{\alpha_2} > -1 \\ -\frac{\alpha_1 + \alpha_2}{\alpha_3} - 1, & \alpha_1 \geq 0, \alpha_2 \geq 0, \alpha_3 < 0, \frac{\alpha_1 + \alpha_2}{\alpha_3} < -1 \\ 0, & \alpha_1 \leq 0, \alpha_2 \leq 0, \alpha_3 > 0, \frac{\alpha_1 + \alpha_2}{\alpha_3} < -1 \\ \min\left\{-\frac{\alpha_2 + \alpha_3}{\alpha_1}, -\frac{\alpha_1 + \alpha_3}{\alpha_2}\right\} - 1, & \alpha_1 < 0, \alpha_2 < 0, \alpha_3 > 0, \frac{\alpha_1 + \alpha_3}{\alpha_2} > -1 \\ 0, & \alpha_1 > 0, \alpha_2 > 0, \alpha_3 < 0, \frac{\alpha_1 + \alpha_2}{\alpha_3} > -1. \end{cases} \quad (11)$$

and

$$F^-(\boldsymbol{\alpha}) = F^+(-\boldsymbol{\alpha}).$$

For a heteroclinic cycle, we denote by  $\sigma_j$  the local stability index along the trajectory leading to the equilibrium  $\xi_j$ . We reproduce a result from [11]

which determines the stability indices of heteroclinic cycles such as those in the RSP game.

**Theorem 5.1** (Theorem 3.11 in [11]). *Let  $M_j$  be transition matrices of a collection of maps associated with a heteroclinic cycle. Assume at least one  $j \in \{1, \dots, m\}$  is such that at least one entry of  $M_j$  is negative.*

- (a) *If, for at least one  $j$ , the matrix  $M^{(j)}$  does not satisfy conditions (i)-(iii) of Lemma 4.1, then  $\sigma_j = -\infty$  for all  $j = 1, \dots, m$  and the cycle is not an attractor.*
- (b) *If, for at least one  $j$ , the matrix  $M^{(j)}$  satisfies conditions (i)-(iii) of Lemma 4.1, then the cycle is f.a.s. Furthermore, for each  $j$  there exist vectors  $\alpha_1, \alpha_2, \dots, \alpha_N$ ,  $j = 1, \dots, m$ , such that*

$$\sigma_j = \min_{i=1, \dots, N} \{F^{index}(\alpha_i)\}.$$

Applying Theorem 5.1 to the  $C_0$ -cycle gives the stability indices of both connections in the cycle.

**Theorem 5.2.** *For the  $C_0$ -cycle of the RSP game,*

- (a) *if  $\varepsilon_x + \varepsilon_y > 0$ , then all stability indices are  $-\infty$  and the cycle is not an attractor.*
- (b) *if  $\varepsilon_x + \varepsilon_y < 0$ , then the stability indices are*

$$\sigma_0 = \frac{1 - \varepsilon_x}{1 + \varepsilon_x} > 0, \quad \sigma_1 = \frac{1 - \varepsilon_y}{1 + \varepsilon_y} > 0,$$

*and the cycle is e.a.s.*

*Proof.* The proof proceeds in two steps: first, we establish that Lemma 4.1 holds if and only if  $\varepsilon_x + \varepsilon_y < 0$ . From Theorem 5.1(a) this is enough to prove part (a) in this theorem. Then, when  $\varepsilon_x + \varepsilon_y < 0$ , we use the function  $F^{index}$  to calculate the stability index for each connection in the  $C_0$ -cycle. Since both stability indices are positive, the heteroclinic cycle is e.a.s. by virtue of Theorem 2.5.

**Step 1:** Consider the transition matrices  $M_0^{(0)}$  and  $M_0^{(1)}$  in (9) associated with the  $C_0$ -cycle. According to Theorem 5.1, we have to check conditions (i)-(iii) of Lemma 4.1 for only one of these matrices in view of similarity.

Let  $M \equiv M_0^{(0)}$ . The eigenvalues of  $M$  are the roots  $\lambda_1, \lambda_2, \lambda_3 \in \mathbb{C}$  of the characteristic polynomial  $p(\lambda) = \det(M - \lambda I) = 0$  where

$$p(\lambda) = -\lambda^3 + \operatorname{tr}(M)\lambda^2 - B(M)\lambda + 1 \quad (12)$$

with

$$\operatorname{tr}(M) \equiv \lambda_1 + \lambda_2 + \lambda_3 = \frac{-3 - 3\varepsilon_x - 3\varepsilon_y + \varepsilon_x\varepsilon_y}{4}, \quad (13)$$

$$B(M) \equiv \lambda_1\lambda_2 + \lambda_1\lambda_3 + \lambda_2\lambda_3 = \frac{-3 + 3\varepsilon_x + 3\varepsilon_y + \varepsilon_x\varepsilon_y}{4},$$

$$\det(M) \equiv \lambda_1\lambda_2\lambda_3 = 1. \quad (14)$$

First, we observe that  $\operatorname{tr}(M)$  and  $B(M)$ , as functions of  $(\varepsilon_x, \varepsilon_y)$ , are bounded on  $] -1, 1[ \times ] -1, 1[$  such that  $-2 < \operatorname{tr}(M) < 1$  and  $-2 < B(M) < 1$ , as well as the product  $\operatorname{tr}(M)B(M) = \frac{1}{16}(3 - \varepsilon_x\varepsilon_y)^2 - \frac{9}{16}(\varepsilon_x + \varepsilon_y)^2$ , with

$$-2 < \operatorname{tr}(M)B(M) < 1. \quad (15)$$

Furthermore,  $\operatorname{tr}(M) + B(M) = -\frac{1}{2}(3 - \varepsilon_x\varepsilon_y) < 0$ , for all  $-1 < \varepsilon_x, \varepsilon_y < 1$ , which implies that  $\operatorname{tr}(M) < 0$  or  $B(M) < 0$ .

By [7, p. 33], the discriminant of the real cubic polynomial (12) is

$$\begin{aligned} \Delta(\varepsilon_x, \varepsilon_y) &= 18\operatorname{tr}(M)B(M) - 4\operatorname{tr}(M)^3 + \operatorname{tr}(M)^2B(M)^2 - 4B(M)^3 - 27 \\ &= \frac{1}{256} \left[ (\varepsilon_x^2 - 9)^2 \varepsilon_y^4 + (-80\varepsilon_x^3 - 432\varepsilon_x) \varepsilon_y^3 + (-18\varepsilon_x^4 - 396\varepsilon_x^2 - 162) \varepsilon_y^2 \right. \\ &\quad \left. + (-432\varepsilon_x^3 - 3024\varepsilon_x) \varepsilon_y + 81\varepsilon_x^4 - 162\varepsilon_x^2 - 3375 \right]. \end{aligned}$$

For each  $-1 < \varepsilon_x < 1$ ,  $\Delta(\varepsilon_x, \cdot)$  can be regarded as a quartic polynomial in  $\varepsilon_y$  and, computing the new discriminant, we obtain

$$\Delta(\varepsilon_x) = -\frac{59049}{67108864} (\varepsilon_x^2 + 15)^3 (\varepsilon_x^2 + 3)^8$$

so that  $\Delta(\varepsilon_x) < 0$ , for all  $-1 < \varepsilon_x < 1$ . From [25], it means that, for each  $-1 < \varepsilon_x < 1$ , the polynomial  $\Delta(\varepsilon_x, \cdot)$  has four distinct roots where two of them are real and the remaining two are complex conjugates. Notice that

$$\begin{aligned} \Delta(\varepsilon_x, -1) &= -\frac{1}{4}(1 - \varepsilon_x)(\varepsilon_x^3 + 9\varepsilon_x^2 + 54) < 0 \\ \Delta(\varepsilon_x, 0) &= \frac{27}{256}(3\varepsilon_x^4 - 6\varepsilon_x^2 - 125) < 0 \\ \Delta(\varepsilon_x, 1) &= -\frac{1}{4}(1 + \varepsilon_x)(-\varepsilon_x^3 + 9\varepsilon_x^2 + 54) < 0, \end{aligned}$$

for all  $-1 < \varepsilon_x < 1$ . Actually, the graph of  $\Delta(\varepsilon_x, \cdot)$  as a function of  $\varepsilon_y$  behaves asymptotically like its leading term  $\frac{1}{256}(\varepsilon_x^2 - 9)^2 \varepsilon_y^4$ . Since this is a four-degree monomial in  $\varepsilon_y$  with a positive coefficient  $\frac{1}{256}(\varepsilon_x^2 - 9)^2$ , then  $\Delta(\varepsilon_x, \cdot)$  becomes increasingly large, whenever  $\varepsilon_y$  is large approaching either  $-\infty$  or  $+\infty$ . Under these circumstances,  $\Delta(\varepsilon_x, \varepsilon_y) < 0$ , for all  $-1 < \varepsilon_x, \varepsilon_y < 1$ , regardless the number of turning points (one or three). Hence, the cubic characteristic polynomial  $p(\lambda)$  has one real root and two complex conjugates roots.

Suppose  $\lambda_1 \in \mathbb{R}$  and  $\lambda_{2,3} = \alpha \pm \beta i \in \mathbb{C}$ , with  $\alpha \in \mathbb{R}$  and  $\beta > 0$ . Immediately, condition (i) holds true if  $\lambda_{\max} = \lambda_1$ . In turn, equalities (14) and (13) lead to

$$\begin{aligned} \lambda_1(\alpha + \beta i)(\alpha - \beta i) = 1 &\Leftrightarrow \lambda_1(\alpha^2 + \beta^2) = 1 \Leftrightarrow \lambda_1 = \frac{1}{\alpha^2 + \beta^2}, \\ \text{tr}(M) = \lambda_1 + \alpha + \beta i + \alpha - \beta i &\Leftrightarrow \text{tr}(M) = \lambda_1 + 2\alpha \Leftrightarrow \lambda_1 = \text{tr}(M) - 2\alpha. \end{aligned} \quad (16)$$

From here, assuming that condition (ii) is satisfied,  $\lambda_1 > 1$  yields  $\alpha^2 + \beta^2 < 1$  whence  $-1 < \alpha, \beta < 1$  and  $\text{tr}(M) > -1$ . It follows that  $B(M) < 0$ , for every  $-1 < \varepsilon_x, \varepsilon_y < 1$ . In other words,

$$-1 < \text{tr}(M) < 1, \quad -2 < B(M) < 0, \quad (17)$$

are necessary conditions for  $\lambda_1 > 1$ . By virtue of Cardano's formula, the real algebraic solution of cubic equation (12) is

$$\lambda_1 = R + S + \frac{\text{tr}(M)}{3}$$

where

$$R = \sqrt[3]{-\frac{b}{2} + \sqrt{\left(\frac{b}{2}\right)^2 + \left(\frac{a}{3}\right)^3}}, \quad S = \sqrt[3]{-\frac{b}{2} - \sqrt{\left(\frac{b}{2}\right)^2 + \left(\frac{a}{3}\right)^3}}$$

and

$$a = \frac{3B(M) - (\text{tr}(M))^2}{3}, \quad b = \frac{-2(\text{tr}(M))^3 + 9\text{tr}(M)B(M) - 27}{27},$$

with  $\text{sign}\left(\left(\frac{b}{2}\right)^2 + \left(\frac{a}{3}\right)^3\right) = -\text{sign}(\Delta(\varepsilon_x, \varepsilon_y))$ . Because of (17) and (15), we have that  $a < 0$  and  $b < 0$ , for all  $-1 < \varepsilon_x, \varepsilon_y < 1$ . Then, both  $R$  and  $S$  are

positive, for all  $-1 < \varepsilon_x, \varepsilon_y < 1$ . Now, we determine under what conditions  $\lambda_1 > 1$  is true. So,

$$\lambda_1 = R + S + \frac{\text{tr}(M)}{3} > 1 \Leftrightarrow R + S > 1 - \frac{\text{tr}(M)}{3}.$$

Due to the presence of cubic roots, we raise to power three and obtain

$$\begin{aligned} (R + S)^3 > \left(1 - \frac{\text{tr}(M)}{3}\right)^3 &\Leftrightarrow R^3 + 3RS(R + S) + S^3 > \left(1 - \frac{\text{tr}(M)}{3}\right)^3 \\ &\Leftrightarrow 3\left(-\frac{a}{3}\right)(R + S) - b > \left(1 - \frac{\text{tr}(M)}{3}\right)^3 \\ &\Leftrightarrow R + S > -\frac{1}{a} \left[ \left(1 - \frac{\text{tr}(M)}{3}\right)^3 + b \right]. \end{aligned}$$

We deduce that  $\lambda_1 > 1$  if and only if

$$1 - \frac{\text{tr}(M)}{3} > -\frac{1}{a} \left[ \left(1 - \frac{\text{tr}(M)}{3}\right)^3 + b \right] \Leftrightarrow -\frac{3}{2}(\varepsilon_x + \varepsilon_y) > 0 \Leftrightarrow \varepsilon_x + \varepsilon_y < 0.$$

Finally, we show that (iii) is valid directly, provided that both (i) and (ii) hold true. Indeed,  $M$  (and  $M_0^{(1)}$  also) is a  $3 \times 3$  real matrix of the form

$$\begin{bmatrix} m_{11} & m_{12} & 1 \\ m_{21} & m_{22} & 0 \\ m_{31} & 1 & 0 \end{bmatrix} \quad (18)$$

such that any eigenvector associated with an eigenvalue  $\lambda \in \mathbb{R}$  is a scalar multiple of

$$\mathbf{w} = \begin{bmatrix} \lambda - m_{22} \\ m_{21} \\ (\lambda - m_{11})(\lambda - m_{22}) - m_{12}m_{21} \end{bmatrix}.$$

For  $\lambda = \lambda_{\max}$  and  $\mathbf{w} = \mathbf{w}^{\max}$ , condition (iii) is fulfilled by  $M$  as long as all the components of  $\mathbf{w}^{\max}$  have the same sign. Assume that  $M$  satisfies (i) and (ii) with  $\lambda_{\max} = \lambda_1$  and  $\mathbf{w}^{\max} = \mathbf{w}_1 = (w_{11}, w_{12}, w_{13})$ . Then, we have

$$w_{11} = \lambda_1 + \frac{1 + \varepsilon_y}{2} > 0, \quad w_{12} = \frac{3 + \varepsilon_y^2}{4} > 0,$$

for all  $-1 < \varepsilon_x, \varepsilon_y < 1$ . On other hand, from  $p(\lambda_1) = 0$  comes out  $\lambda_1(\lambda_1^2 - \text{tr}(M)\lambda_1 + B(M)) = 1$ . Since  $\lambda_1$  is real and greater than one, it follows that  $0 < \lambda_1^2 - \text{tr}(M)\lambda_1 + B(M) < 1$  and

$$w_{13} = \lambda_1^2 - \text{tr}(M)\lambda_1 + B(M) + \frac{1 - \varepsilon_y}{2} > 0,$$

for all  $-1 < \varepsilon_x, \varepsilon_y < 1$ .

In short,  $M$  satisfies (i)-(iii) if and only if  $\varepsilon_x + \varepsilon_y < 0$ . Both transition matrices  $M_{101}$  and  $M_{010}$  in (8) involve negative entries and Theorem 5.1 guarantees that the  $C_0$ -cycle is f.a.s. which is equivalent to  $\ell(U^{-\infty}(M^{(j)})) > 0$ ,  $j = 0, 1$ , by definition.

**Step 2:** In order to simplify notation, let  $(w, z_1, z_2) = (x_1, x_2, x_3) \equiv \mathbf{x}$  and  $\boldsymbol{\eta} = (\ln(x_1), \ln(x_2), \ln(x_3))$ . By virtue of (10), the local basin of attraction  $\mathcal{B}_\delta^{\pi_0}$  for points  $\mathbf{x}$  near  $\xi_0$  is defined through the sequences  $(M_0^{(0)})^n \boldsymbol{\eta}$  and  $M_{101} (M_0^{(0)})^n \boldsymbol{\eta}$ , for all  $n \geq 0$ . In the same way, the local basin of attraction  $\mathcal{B}_\delta^{\pi_1}$  for points  $\mathbf{x}$  near  $\xi_1$  is defined through the sequences  $(M_0^{(1)})^n \boldsymbol{\eta}$  and  $M_{010} (M_0^{(1)})^n \boldsymbol{\eta}$ , for all  $n \geq 0$ . Taking into account the constructive proof of Theorem 5.1 (see pp. 19–20 in [11]), we have

$$\begin{aligned} \sigma_0 &= \min \left\{ F^{\text{index}}(\mathbf{v}_{\max}^{(0)}), F^{\text{index}}\left(\frac{1-\varepsilon_y}{2}, 1, 0\right), F^{\text{index}}\left(-\frac{1+\varepsilon_x}{2}, 0, 1\right), F^{\text{index}}(1, 0, 0) \right\} \\ \sigma_1 &= \min \left\{ F^{\text{index}}(\mathbf{v}_{\max}^{(1)}), F^{\text{index}}\left(\frac{1-\varepsilon_x}{2}, 1, 0\right), F^{\text{index}}\left(-\frac{1+\varepsilon_y}{2}, 0, 1\right), F^{\text{index}}(1, 0, 0) \right\}, \end{aligned} \quad (19)$$

where  $\mathbf{v}_{\max}^{(j)} = \mathbf{v}_1^{(j)} = (v_{11}^{(j)}, v_{12}^{(j)}, v_{13}^{(j)})$ ,  $j = 0, 1$ , is the line corresponding to the position associated with  $\lambda_{\max}$  of the change of basis matrix from the basis of eigenvectors for  $M_0^{(j)}$  to the canonical basis in  $\mathbb{R}^3$ . We claim that  $F^{\text{index}}(\mathbf{v}_1^{(j)}) = +\infty$ , for every  $j = 0, 1$ . From (11), it means that the components of  $\mathbf{v}_1^{(j)}$  are all non-negative. In general, an eigenvector of a matrix (18) associated with a complex eigenvalue  $\mu + \omega i$  is a scalar multiple of

$$\mathbf{u} = \begin{bmatrix} \mu - m_{22} \\ m_{21} \\ (\mu - m_{11})(\mu - m_{22}) - \omega^2 - m_{12}m_{21} \end{bmatrix} + i \begin{bmatrix} \omega \\ 0 \\ (2\mu - m_{11} - m_{22})\omega \end{bmatrix}.$$

As a consequence,  $\{\mathbf{w}, \text{Im}(\mathbf{u}), \text{Re}(\mathbf{u})\}$  is a basis of eigenvectors for (18). Simple algebra attests that

$$\mathbf{v}_1 = \begin{bmatrix} \frac{m_{11} + m_{22} - 2\mu}{(\mu - \lambda)^2 + \omega^2} & \frac{(\mu - m_{22})^2 + \omega^2 + m_{12}m_{21}}{m_{21}((\mu - \lambda)^2 + \omega^2)} & \frac{1}{m_{21}((\mu - \lambda)^2 + \omega^2)} \end{bmatrix}.$$

Specifically for similar matrices  $M \equiv M_0^{(0)}$  and  $M_0^{(1)}$ ,  $m_{11} + m_{22} = \text{tr}(M) = \text{tr}(M_0^{(1)})$ ,  $\lambda = \lambda_1$  and  $\mu + \omega i = \alpha + \beta i$ . Due to (16),  $m_{11} + m_{22} - 2\mu =$



$\text{tr}(M) - 2\alpha = \lambda_1$ . Moreover, entries  $m_{12}$  and  $m_{21}$  are positive in both matrices (9). Hence, for every  $j = 0, 1$ ,  $\mathbf{v}_1^{(j)}$  have surely non-negative components.

Since  $-1 < \varepsilon_x, \varepsilon_y < 1$  implies  $\frac{1}{-\frac{1-\varepsilon_x}{2}} < -1$  and  $\frac{1}{-\frac{1-\varepsilon_y}{2}} < -1$ , we simplify local stability indices in (19) through (11) such that the minima are

$$\begin{aligned}\sigma_0 &= F^{\text{index}}\left(-\frac{1+\varepsilon_x}{2}, 0, 1\right) = -\frac{1}{-\frac{1-\varepsilon_x}{2}} - 1 = \frac{1-\varepsilon_x}{1+\varepsilon_x} > 0 \\ \sigma_1 &= F^{\text{index}}\left(-\frac{1+\varepsilon_y}{2}, 0, 1\right) = -\frac{1}{-\frac{1-\varepsilon_y}{2}} - 1 = \frac{1-\varepsilon_y}{1+\varepsilon_y} > 0.\end{aligned}$$

□

The stability of the remaining heteroclinic cycles in the quotient network of the RSP game is given in Theorems 5.3–5.6. The proofs are omitted because they replicate the proof of Theorem 5.2 itself by using the transition matrices provided in Appendix C.

**Theorem 5.3.** *For the  $C_1$ -cycle of the RSP game,*

- (a) *if either  $\varepsilon_x + \varepsilon_y < 0$ , or  $(5 - \varepsilon_x)\varepsilon_y^2 + (\varepsilon_x^2 + 10\varepsilon_x + 1)\varepsilon_y - (1 - \varepsilon_x)(4 + 5\varepsilon_x) < 0$ , or  $\varepsilon_x - \varepsilon_y > 0$ , then the cycle is not an attractor and all stability indices are  $-\infty$ .*
- (b) *if  $\varepsilon_x + \varepsilon_y > 0$ , and  $(5 - \varepsilon_x)\varepsilon_y^2 + (\varepsilon_x^2 + 10\varepsilon_x + 1)\varepsilon_y - (1 - \varepsilon_x)(4 + 5\varepsilon_x) > 0$ , and  $\varepsilon_x - \varepsilon_y < 0$ , then the stability indices are*

$$\sigma_1 = -\frac{1-\varepsilon_y}{1+\varepsilon_y} < 0, \quad \sigma_2 = \frac{\varepsilon_y - \varepsilon_x}{1 - \varepsilon_y} > 0,$$

*and the cycle is f.a.s.*

**Theorem 5.4.** *For the  $C_2$ -cycle of the RSP game,*

- (a) *if either  $\varepsilon_x + \varepsilon_y < 0$ , or  $(5 + \varepsilon_x)\varepsilon_y^2 + (-\varepsilon_x^2 + 10\varepsilon_x - 1)\varepsilon_x - (1 + \varepsilon_x)(4 - 5\varepsilon_x) < 0$ , or  $\varepsilon_x - \varepsilon_y < 0$ , then the cycle is not an attractor and all stability indices are  $-\infty$ .*
- (b) *if  $\varepsilon_x + \varepsilon_y > 0$ , and  $(5 + \varepsilon_x)\varepsilon_y^2 + (-\varepsilon_x^2 + 10\varepsilon_x - 1)\varepsilon_x - (1 + \varepsilon_x)(4 - 5\varepsilon_x) > 0$ , and  $\varepsilon_x - \varepsilon_y > 0$ , then the stability indices are*

$$\sigma_0 = -\frac{1-\varepsilon_x}{1+\varepsilon_x} < 0, \quad \sigma_2 = \frac{\varepsilon_x - \varepsilon_y}{1 - \varepsilon_x} > 0,$$

*and the cycle is f.a.s.*

**Theorem 5.5.** *For the  $C_3$ -cycle of the RSP game, all stability indices are  $-\infty$ , for every  $-1 < \varepsilon_x, \varepsilon_y < 1$ , and the cycle is not an attractor.*

**Theorem 5.6.** *For the  $C_4$ -cycle of the RSP game, all stability indices are  $-\infty$ , for every  $-1 < \varepsilon_x, \varepsilon_y < 1$ , and the cycle is not an attractor.*

Notice that the heteroclinic cycles  $C_3$  and  $C_4$  never exhibit any kind of stability. This may justify their absence from the numerical observations made by Sato *et al.* [26]. The heteroclinic cycles  $C_1$  and  $C_2$  are never e.a.s., making them difficult to detect in simulations. They are f.a.s. in a subset of the complement of the stability region for the  $C_0$ -cycle. The regions of stability of the cycles  $C_0$ ,  $C_1$ , and  $C_2$  are depicted in Figure 2.

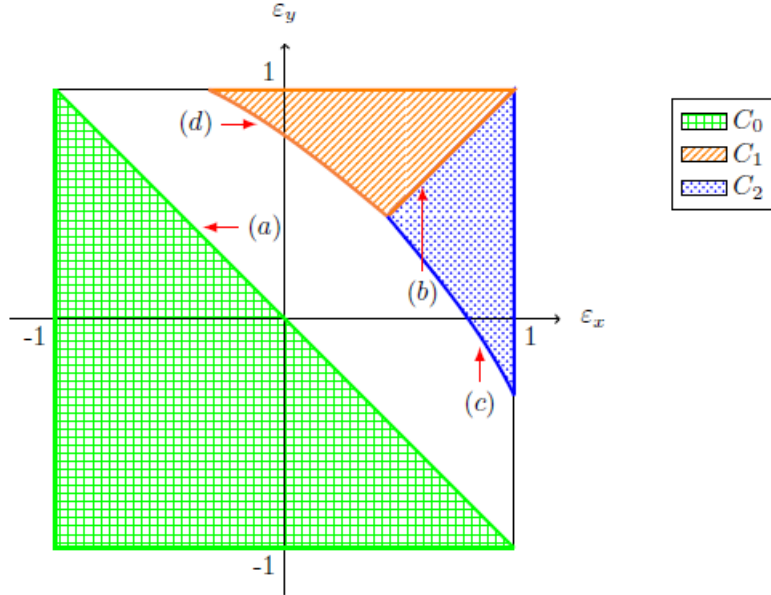


Figure 2: Regions in parameter space where the cycles  $C_0$ ,  $C_1$  and  $C_2$  exhibit some stability. The lines in the figure are: (a)  $\varepsilon_x + \varepsilon_y = 0$ ; (b)  $\varepsilon_x - \varepsilon_y = 0$ ; (c)  $\varepsilon_y = \frac{-(1+10\varepsilon_x+\varepsilon_x^2)+\sqrt{81+24\varepsilon_x-2\varepsilon_x^2+40\varepsilon_x^3+\varepsilon_x^4}}{2(5-\varepsilon_x)}$ ; (d)  $\varepsilon_y = \frac{1-10\varepsilon_x+\varepsilon_x^2+\sqrt{81-24\varepsilon_x-2\varepsilon_x^2-40\varepsilon_x^3+\varepsilon_x^4}}{2(5+\varepsilon_x)}$ .

## 6 Applications

As motivating examples, without the claim to detailed accuracy, we revisit the two hypothetical RSP games with two players suggested in Section 1.

Take then the model of Noel [22] and let the players be a major firm (player 1) and an independent firm (player 2). Changing slightly the actions suggested in the Introduction, let each player choose among

**(R)** fix a low price;

**(S)** fix an intermediate price;

**(P)** fix a high price.

A low price (R) beats an intermediate price (S) because the higher number of buyers compensates for the loss in price. An intermediate price (S) beats a high price (P) for the same reason. A high price (P) beats a low price (R) because, even though there are fewer buyers, the price difference makes up for the reduction in sales.

Assume that consumers buy the cheapest between two consecutive price levels but when faced with a high and a low price, consumers associate quality to the highest price and buy at this price. The cycle  $C_0$  corresponds to the following sequence of outcomes

Firm 1 chooses	Firm 2 chooses	Winner
low price	high price	Firm 2
intermediate price	high price	Firm 1
intermediate price	low price	Firm 2
high price	low price	Firm 1
high price	intermediate price	Firm 2
low price	intermediate price	Firm 1
low price	high price	Firm 2

A first approach to the model of Hopkins and Seymour [14] is to consider aggregate sellers as Player 1 and aggregate consumers as Player 2. The actions for the two groups are as follows, where we use “s” for sellers and “c” for consumers to distinguish among R, S and P,

**Sellers:**

**(R-s)** fix a low price;

**(S-s)** fix an intermediate price;

**(P-s)** fix a high price;

**Consumers:**

(**R-c**) be poorly informed;

(**S-c**) be reasonably informed;

(**P-c**) be very well informed.

Within the sellers, the relation among the possible actions is as suggested above. Within consumers, being very well informed (P-c) beats being poorly informed (R-c) since a deal can be obtained which compensates the investment made to obtain the information. Being reasonably well informed (S-c) beats being very well informed (P-c) since a good enough deal can still be obtained without spending so much on gathering information. Being poorly informed (R-c) beats being reasonably informed (S-c) because poor information is free and there is still a chance of coming across a good deal.

Assume that sellers want to sell at the highest possible price. Assume also that consumers want to buy at low prices and not spend on gathering information if they can avoid it. However, when well informed they will not buy at a very high price. We have the following relations among the possible choices, leading to the cycle  $C_0$ :

- P-c beats R-s because gathering a lot of information puts pressure on the sellers to lower their price;
- S-c beats P-s because a reasonably informed consumer will not buy at a high price;
- R-c beats S-s because the seller could have sold for a higher price to the reasonably informed consumer;
- P-s beats R-c because the goods are sold at a high price;
- S-s beats P-c because the goods are bought anyway;
- R-s beats S-c because consumers spent unnecessarily to gather information.

## 7 Concluding remarks

Making use of recent developments in the study of dynamical systems, particularly in the study of stability of heteroclinic cycles, we classify all the cycles of the RSP game according to their stability as a function of two parameters. These two parameters describe the payoff players receive when the

outcome of their choice of actions is a tie and we allow the payoffs to range from almost as bad as a loss to almost as good as a win. We show that when at least one player has a negative payoff for a tie, low enough that the sum of payoffs for a tie is itself negative, then both players avoid any choice of action leading to a tie. This is a situation that is as stable as possible. The corresponding situation when the sum of payoffs for a tie is positive is not as stable, although it does exhibit some intermediate level of stability. This can be explained by the fact that if the players oscillate between a tie and a win for one (same) player, then the player that never wins is not very happy with the situation and tries to deviate. These results are consistent with numerical simulations and experiments referred throughout the current work and may help to clarify empirical examples of the RSP cycles discovered in nature and economy.

We note that our RSP game is not necessarily zero-sum, taking into account Sigmund's [28] concern that "For most types of social and economic interactions, the assumption that the interests of the two players are always diametrically opposite does not hold." In particular, the payoff matrices reflect asymmetry which is a feature that arises either in interpopulation or intrapopulation interactions. Social and economic dilemmas between consumers and sellers, firms and workers are examples of situations where agents frequently adopt asymmetric positions. Also, differences in access to, and availability of, resources asymmetrically affect individuals' behaviour within each class of agents.

Applications exhibiting cyclic behaviour arising through the existence of a limit cycle, rather than a heteroclinic cycle, are studied by Semmann *et al.* [27] in the context of public goods, as well as by Mobilia [21] and Toupou and Strogatz [31] in the context of populations under different types of mutation. This approach to cyclic behaviour is beyond the scope of the present article.

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## References

- [1] M.A.D. Aguiar and S.B.S.D. Castro (2010) Chaotic switching in a two-person game. *Physica D: Nonlinear Phenomena*, **239** (16), 1598–1609.

- [2] T. Börgers and R. Sarin (1997) Learning Through Reinforcement and Replicator Dynamics. *Journal of Economic Theory*, **77** (1), 1–14.
- [3] W. Brannath (1994) Heteroclinic networks on the tetrahedron. *Nonlinearity*, **7** (5), 1367–1384.
- [4] T.N. Cason, D. Friedman and E. Hopkins (2014) Cycles and Instability in a Rock-Paper-Scissors Population Game: A Continuous Time Experiment. *The Review of Economic Studies*, **81** (1), 112–136.
- [5] T. Cason, D. Friedman and F. Wagener (2005) The Dynamics of Price Dispersion, or Edgeworth Variations. *Journal of Economic Dynamics and Control*, **29** (4), 801–822.
- [6] S.B.S.D. Castro and A. Lohse (2014) Stability in simple heteroclinic networks in  $\mathbb{R}^4$ . *Dynamical Systems*, **29** (4), 451–481.
- [7] L.E. Dickson (1914) *Elementary Theory of Equations*. John Wiley and Sons, New York.
- [8] F.Y. Edgeworth (1925) The pure theory of monopoly. In *Papers relating to political economy*, pages 111–142. Macmillan, London.
- [9] M. Field (1996) *Lectures on bifurcations, dynamics and symmetry*. Pitman Research Notes in Mathematics Series, vol. 356, Longman.
- [10] D. Friedman (1991) Evolutionary games in economics. *Econometrica*, **59** (3), 637–666.
- [11] L. Garrido-da-Silva and S.B.S.D. Castro (2016) Stability of one-dimensional heteroclinic connections. *arXiv:1606.02592*.
- [12] M. Golubitsky, I.N. Stewart, and D.G. Schaefer (1988) *Singularities and groups in bifurcation theory*, vol. 2. Springer-Verlag, New York.
- [13] J. Hofbauer and K. Sigmund (1998) *Evolutionary Games and Population Dynamics*. Cambridge University Press, Cambridge.
- [14] E. Hopkins and R.M. Seymour (2002) The Stability of Price Dispersion under Seller and Consumer Learning. *International Economic Review*, **43** (4), 1157–1190.
- [15] V. Kirk and M. Silber (1994) A competition between heteroclinic cycles. *Nonlinearity*, **7** (6), 1605–1621.

- [16] M. Krupa and I. Melbourne (1995) Asymptotic stability of heteroclinic cycles in systems with symmetry. *Ergodic Theory and Dynamical Systems*, **15**, 121–147.
- [17] M. Krupa and I. Melbourne (2004) Asymptotic stability of heteroclinic cycles in systems with symmetry II. *Proceedings of the Royal Society of Edinburgh, Section: A Mathematics*, **134**, 1177–1197.
- [18] R. Lahkar and R.M. Seymour (2014) The dynamics of generalized reinforcement learning. *Journal of Economic Theory*, **151**, 584–595.
- [19] A. Lohse (2015) Stability of heteroclinic cycles in transverse bifurcations. *Physica D: Nonlinear Phenomena*, **310**, 95–103.
- [20] I. Melbourne (1991) An example of a non-asymptotically stable attractor. *Nonlinearity*, **4** (3), 835–844.
- [21] M. Mobilia (2010) Oscillatory dynamics in rock-paper-scissors games with mutations. *Journal of Theoretical Biology*, **264**, 1–10.
- [22] M.D. Noel (2007) Edgeworth Price Cycles: Evidence from the Toronto Retail Gasoline Market. *The Journal of Industrial Economics*, **55** (1), 69–92.
- [23] O. Podvigina (2012) Stability and bifurcations of heteroclinic cycles of type Z. *Nonlinearity*, **25** (6), 1887–1917.
- [24] O. Podvigina and P. Ashwin (2011) On local attraction properties and a stability index for heteroclinic connections. *Nonlinearity*, **24** (3), 887–929.
- [25] E.L. Rees (1922) Graphical discussion of the roots of a quartic equation. *The American Mathematical Monthly*, **29** (2), 51–55.
- [26] Y. Sato, E. Akiyama and J.P. Crutchfield (2005) Stability and Diversity in Collective Adaptation. *Physica D*, **210** (12), 21–57.
- [27] D. Semmann, H.-J. Krambeck and M. Milinski (2003) Volunteering leads to rock-paper-scissors dynamics in a public goods game. *Nature*, **425**, 390–393.
- [28] K. Sigmund (2011) Introduction to evolutionary game theory. *Proceedings of Symposia in Applied Mathematics*, **69**, 21–57.

- [29] B. Sinervo and C.M. Lively (1996) The rock-scissors-paper game and the evolution of alternative male strategies. *Nature*, **380**, 240–246.
- [30] A. Szolnoki, M. Mobilia, L.L. Jiang, B. Szczesny, A.M. Rucklidge and M. Perc (2014) Cyclic dominance in evolutionary games: A review. *Journal of the Royal Society Interface*, **11** (100).
- [31] D.F.P. Toupou and S.H. Strogatz (2015) Nonlinear dynamics of the rock-paper-scissors game with mutations. *Physical Review E*, **91**, 052907.

## A Partial turn maps

In the usual way, a local map  $\phi_{kjl} : H_j^{in,k} \rightarrow H_j^{out,l}$  ( $j \neq k, l = 0, 1, 2$ ) approximates the flow in a neighbourhood of  $\xi_j$  and a global map  $\psi_{jl} : H_j^{out,l} \rightarrow H_l^{in,j}$  approximates the flow near connection  $[\xi_j \rightarrow \xi_l]$ . Composing the local and global maps together gives  $\tilde{g}_{kjl} : H_j^{in,k} \rightarrow H_l^{in,j}$ , taking points from an incoming connection at  $\xi_j$  to an outgoing connection at  $\xi_l$ , along the connection  $[\xi_j \rightarrow \xi_l]$ . Only the components  $w, z_1, z_2$  of  $\tilde{g}_{kjl}$  are relevant for the stability of a heteroclinic cycle. We define the restriction of the map  $\tilde{g}_{kjl}$  into the subspace  $Q_{kj}^\perp = \{(0, 0, 0, w, z_1, z_2) : w, z_1, z_2 \in \mathbb{R}\}$  as  $g_{kjl} : \hat{H}_j^{in,k} \rightarrow \hat{H}_l^{in,j}$  where  $\hat{H}_j^{in,k} = H_j^{in,k} \cap Q_{kj}^\perp \cong \mathbb{R}^3$ . We call  $g_{kjl}$  a partial turn map.

**$C_1$ -cycle:**

$$g_{212} : \hat{H}_1^{in,2} \rightarrow \hat{H}_2^{in,1}, \quad g_{212}(w, z_1, z_2) = \left( z_1 w^{\frac{2}{1+\varepsilon_y}}, w^{\frac{1-\varepsilon_x}{1+\varepsilon_y}}, z_2 w^{-\frac{2}{1+\varepsilon_y}} \right), \quad \text{for } 0 \leq z_2 < w^{\frac{2}{1+\varepsilon_y}}$$

$$g_{121} : \hat{H}_2^{in,1} \rightarrow \hat{H}_1^{in,2}, \quad g_{121}(w, z_1, z_2) = \left( z_2 w^{-\frac{1-\varepsilon_y}{1-\varepsilon_x}}, z_1 w^{\frac{1+\varepsilon_x}{1-\varepsilon_x}}, w^{\frac{1+\varepsilon_y}{1-\varepsilon_x}} \right), \quad \text{for } 0 \leq z_2 < w^{\frac{1-\varepsilon_y}{1-\varepsilon_x}}.$$

**$C_2$ -cycle:**

$$g_{202} : \hat{H}_0^{in,2} \rightarrow \hat{H}_2^{in,0}, \quad g_{202}(w, z_1, z_2) = \left( z_1 w^{\frac{2}{1+\varepsilon_x}}, w^{\frac{1-\varepsilon_y}{1+\varepsilon_x}}, z_2 w^{-\frac{2}{1+\varepsilon_x}} \right), \quad \text{for } 0 \leq z_2 < w^{\frac{2}{1+\varepsilon_x}}$$

$$g_{020} : \hat{H}_2^{in,0} \rightarrow \hat{H}_0^{in,2}, \quad g_{020}(w, z_1, z_2) = \left( z_2 w^{-\frac{1-\varepsilon_x}{1-\varepsilon_y}}, z_1 w^{\frac{1+\varepsilon_y}{1-\varepsilon_y}}, w^{\frac{1+\varepsilon_x}{1-\varepsilon_y}} \right), \quad \text{for } 0 \leq z_2 < w^{\frac{1-\varepsilon_x}{1-\varepsilon_y}}.$$



**$C_3$ -cycle:**

$$\begin{aligned}
g_{201} : \hat{H}_0^{in,2} &\rightarrow \hat{H}_1^{in,0}, & g_{201}(w, z_1, z_2) &= \left( z_1 w, z_2 w^{-\frac{1+\varepsilon_x}{2}}, w^{\frac{1+\varepsilon_y}{2}} \right), & \text{for } 0 \leq z_2 < w^{\frac{1+\varepsilon_x}{2}} \\
g_{012} : \hat{H}_1^{in,0} &\rightarrow \hat{H}_2^{in,1}, & g_{012}(w, z_1, z_2) &= \left( z_2 w^{-\frac{2}{1+\varepsilon_y}}, z_1 w^{\frac{1-\varepsilon_x}{1+\varepsilon_y}}, w^{\frac{2}{1+\varepsilon_y}} \right), & \text{for } 0 \leq z_2 < w^{\frac{2}{1+\varepsilon_y}} \\
g_{120} : \hat{H}_2^{in,1} &\rightarrow \hat{H}_0^{in,2}, & g_{120}(w, z_1, z_2) &= \left( z_1 w^{\frac{1+\varepsilon_x}{1-\varepsilon_y}}, w^{\frac{1+\varepsilon_y}{1-\varepsilon_y}}, z_2 w^{-\frac{1-\varepsilon_x}{1-\varepsilon_y}} \right), & \text{for } 0 \leq z_2 < w^{\frac{1-\varepsilon_x}{1-\varepsilon_y}}.
\end{aligned}$$

**$C_4$ -cycle:**

$$\begin{aligned}
g_{102} : \hat{H}_0^{in,1} &\rightarrow \hat{H}_2^{in,0}, & g_{102}(w, z_1, z_2) &= \left( z_2 w^{-\frac{2}{1+\varepsilon_x}}, z_1 w^{\frac{1-\varepsilon_y}{1+\varepsilon_x}}, w^{\frac{2}{1+\varepsilon_x}} \right), & \text{for } 0 \leq z_2 < w^{\frac{2}{1+\varepsilon_x}} \\
g_{210} : \hat{H}_1^{in,2} &\rightarrow \hat{H}_2^{in,0}, & g_{210}(w, z_1, z_2) &= \left( z_1 w, z_2 w^{-\frac{1+\varepsilon_y}{2}}, w^{\frac{1+\varepsilon_x}{2}} \right), & \text{for } 0 \leq z_2 < w^{\frac{1+\varepsilon_y}{2}} \\
g_{021} : \hat{H}_2^{in,0} &\rightarrow \hat{H}_0^{in,1}, & g_{021}(w, z_1, z_2) &= \left( z_1 w^{\frac{1+\varepsilon_y}{1-\varepsilon_x}}, w^{\frac{1+\varepsilon_x}{1-\varepsilon_x}}, z_2 w^{-\frac{1-\varepsilon_y}{1-\varepsilon_x}} \right), & \text{for } 0 \leq z_2 < w^{\frac{1-\varepsilon_y}{1-\varepsilon_x}}.
\end{aligned}$$

## B Reduced Poincaré maps

We model the behaviour of trajectories along a heteroclinic cycle through Poincaré maps  $\tilde{\pi}_j : H_j^{in,l} \rightarrow H_j^{in,l}$  ( $j \neq l$ ) obtained via the composition of local and global maps in the correct order. Restricting these to  $(w, z_1, z_2)$  coordinates or equivalently into  $Q_{jl}^\perp$ , we have  $\pi_j : \hat{H}_j^{in,l} \rightarrow \hat{H}_j^{in,l}$  where  $\hat{H}_j^{in,l} = H_j^{in,l} \cap Q_{jl}^\perp \cong \mathbb{R}^3$ . Denote by  $\pi_{j,i}$  a reduced Poincaré map associated with the  $C_i$ -cycle,  $i = 0, 1, 2, 3, 4$ .

**$C_1$ -cycle:**

$$\begin{aligned}
\pi_{1,1} &= g_{121} \circ g_{212} : \hat{H}_1^{in,2} \rightarrow \hat{H}_1^{in,2}, \\
\pi_{1,1}(w, z_1, z_2) &= \left( z_2 z_1^{-\frac{1-\varepsilon_y}{1-\varepsilon_x}} w^{-\frac{2(2-\varepsilon_x-\varepsilon_y)}{(1-\varepsilon_x)(1+\varepsilon_y)}}, z_1^{\frac{1+\varepsilon_x}{1-\varepsilon_x}} w^{\frac{3+\varepsilon_x^2}{(1-\varepsilon_x)(1+\varepsilon_y)}}, z_1^{\frac{1+\varepsilon_y}{1-\varepsilon_x}} w^{\frac{2}{1-\varepsilon_x}} \right) \\
\text{for } 0 \leq z_2 &< \min \left\{ w^{\frac{2}{1+\varepsilon_y}}, z_1^{\frac{1-\varepsilon_y}{1-\varepsilon_x}} w^{\frac{2(2-\varepsilon_x-\varepsilon_y)}{(1-\varepsilon_x)(1+\varepsilon_y)}} \right\} \\
\pi_{2,1} &= g_{212} \circ g_{121} : \hat{H}_2^{in,1} \rightarrow \hat{H}_2^{in,1}, \\
\pi_{2,1}(w, z_1, z_2) &= \left( z_1 z_2^{\frac{2}{1+\varepsilon_y}} w^{\frac{-1+\varepsilon_x+3\varepsilon_y+\varepsilon_x\varepsilon_y}{(1-\varepsilon_x)(1+\varepsilon_y)}}, z_2^{\frac{1-\varepsilon_x}{1+\varepsilon_y}} w^{-\frac{1-\varepsilon_y}{1+\varepsilon_y}}, z_2^{-\frac{2}{1+\varepsilon_y}} w^{\frac{3+\varepsilon_y^2}{(1-\varepsilon_x)(1+\varepsilon_y)}} \right) \\
\text{for } w^{\frac{3+\varepsilon_y^2}{2(1-\varepsilon_x)}} &< z_2 < w^{\frac{1-\varepsilon_y}{1-\varepsilon_x}}.
\end{aligned}$$

**$C_2$ -cycle:**

$$\begin{aligned}
\pi_{0,2} &= g_{020} \circ g_{202} : \hat{H}_0^{in,1} \rightarrow \hat{H}_0^{in,1}, \\
\pi_{0,2}(w, z_1, z_2) &= \left( z_2 z_1^{-\frac{1-\varepsilon_x}{1-\varepsilon_y}} w^{-\frac{2(2-\varepsilon_x-\varepsilon_y)}{(1+\varepsilon_x)(1-\varepsilon_y)}}, z_1^{\frac{1+\varepsilon_y}{1-\varepsilon_y}} w^{\frac{3+\varepsilon_y^2}{(1+\varepsilon_x)(1-\varepsilon_y)}}, z_1^{\frac{1+\varepsilon_x}{1-\varepsilon_y}} w^{\frac{2}{1-\varepsilon_y}} \right) \\
\text{for } 0 \leq z_2 &< \min \left\{ w^{\frac{2}{1+\varepsilon_x}}, z_1^{\frac{1-\varepsilon_x}{1-\varepsilon_y}} w^{\frac{2(2-\varepsilon_x-\varepsilon_y)}{(1+\varepsilon_x)(1-\varepsilon_y)}} \right\} \\
\pi_{2,2} &= g_{202} \circ g_{020} : \hat{H}_1^{in,0} \rightarrow \hat{H}_1^{in,0}, \\
\pi_{2,2}(w, z_1, z_2) &= \left( z_1 z_2^{\frac{2}{1+\varepsilon_x}} w^{\frac{-1+3\varepsilon_x+\varepsilon_y+\varepsilon_x\varepsilon_y}{(1+\varepsilon_x)(1-\varepsilon_y)}}, z_2^{\frac{1-\varepsilon_y}{1+\varepsilon_x}} w^{-\frac{1-\varepsilon_x}{1+\varepsilon_x}}, z_2^{-\frac{2}{1+\varepsilon_x}} w^{\frac{3+\varepsilon_x^2}{(1+\varepsilon_x)(1-\varepsilon_y)}} \right) \\
\text{for } w^{\frac{3+\varepsilon_x^2}{2(1-\varepsilon_y)}} &\leq z_2 < w^{\frac{1-\varepsilon_x}{1-\varepsilon_y}}.
\end{aligned}$$

**$C_3$ -cycle:**

$$\pi_{0,3} = g_{120} \circ g_{012} \circ g_{201} : \hat{H}_0^{in,2} \rightarrow \hat{H}_0^{in,2},$$

$$\pi_{0,3}(w, z_1, z_2) = \left( z_2 z_1^{\frac{-1-3\varepsilon_x-\varepsilon_y+\varepsilon_x\varepsilon_y}{1-\varepsilon_y^2}} w^{\frac{-1-3\varepsilon_x-\varepsilon_y+\varepsilon_x\varepsilon_y}{1-\varepsilon_y^2}}, z_1^{-\frac{2}{1-\varepsilon_y}} w^{-\frac{3+\varepsilon_y^2}{2(1-\varepsilon_y)}}, z_1^{\frac{2(2-\varepsilon_x-\varepsilon_y)}{1-\varepsilon_y^2}} w^{\frac{7-3\varepsilon_x-4\varepsilon_y+\varepsilon_y^2-\varepsilon_x\varepsilon_y^2}{2(1-\varepsilon_y^2)}} \right)$$

$$\text{for } 0 \leq z_2 < w^{\frac{1+\varepsilon_x}{2}} \wedge w^{\frac{-3+2\varepsilon_y+\varepsilon_y^2}{4}} < z_1 < w^{-\frac{7-3\varepsilon_x-4\varepsilon_y+\varepsilon_y^2-\varepsilon_x\varepsilon_y^2}{4(2-\varepsilon_x-\varepsilon_y)}}$$

$$\pi_{1,3} = g_{201} \circ g_{120} \circ g_{012} : \hat{H}_1^{in,0} \rightarrow \hat{H}_1^{in,0},$$

$$\pi_{1,3}(w, z_1, z_2) = \left( z_1 z_2^{\frac{2+\varepsilon_x+\varepsilon_y}{1-\varepsilon_y}} w^{\frac{-3-3\varepsilon_x-3\varepsilon_y+\varepsilon_x\varepsilon_y}{1-\varepsilon_y^2}}, z_1^{-\frac{1+\varepsilon_x}{2}} z_2^{-\frac{3+\varepsilon_x^2}{2(1-\varepsilon_y)}} w^{\frac{9-3\varepsilon_y+3\varepsilon_x^2-\varepsilon_x^2\varepsilon_y}{2(1-\varepsilon_y^2)}}, z_1^{\frac{1-\varepsilon_y}{2}} z_2^{\frac{1+\varepsilon_x}{2}} w^{\frac{-1-3\varepsilon_x-\varepsilon_y+\varepsilon_x\varepsilon_y}{2(1+\varepsilon_y)}} \right)$$

$$\text{for } w^{\frac{2(2-\varepsilon_x-\varepsilon_y)}{(1-\varepsilon_x)(1+\varepsilon_y)}} < z_2 < w^{\frac{2}{1+\varepsilon_y}} \wedge z_1 > z_2^{-\frac{3+\varepsilon_x^2}{(1+\varepsilon_x)(1-\varepsilon_y)}} w^{\frac{9-3\varepsilon_y+3\varepsilon_x^2-\varepsilon_x^2\varepsilon_y}{(1+\varepsilon_x)(1-\varepsilon_y^2)}}$$

$$\pi_{2,3} = g_{012} \circ g_{201} \circ g_{120} : \hat{H}_2^{in,1} \rightarrow \hat{H}_2^{in,1},$$

$$\pi_{2,3}(w, z_1, z_2) = \left( z_1^{-\frac{3+\varepsilon_y^2}{2(1+\varepsilon_y)}} w^{-\frac{7+3\varepsilon_x+4\varepsilon_y+\varepsilon_y^2+\varepsilon_x\varepsilon_y^2}{2(1-\varepsilon_y^2)}}, z_2 z_1^{\frac{1-3\varepsilon_x-\varepsilon_y-\varepsilon_x\varepsilon_y}{2(1+\varepsilon_y)}} w^{\frac{(1+\varepsilon_x)(1-3\varepsilon_x-\varepsilon_y-\varepsilon_x\varepsilon_y)}{2(1-\varepsilon_y^2)}}, z_1^{\frac{2}{1+\varepsilon_y}} w^{\frac{2(2+\varepsilon_x+\varepsilon_y)}{1-\varepsilon_y^2}} \right)$$

$$\text{for } 0 \leq z_2 < \min \left\{ w^{\frac{1-\varepsilon_x}{1-\varepsilon_y}}, z_1^{\frac{1+\varepsilon_x}{2}} w^{\frac{3+\varepsilon_x^2}{2(1-\varepsilon_y)}} \right\} \wedge z_1 > w^{-\frac{7+3\varepsilon_x+4\varepsilon_y+\varepsilon_y^2+\varepsilon_x\varepsilon_y^2}{(3+\varepsilon_y^2)(1-\varepsilon_y)}}$$

**$C_4$ -cycle:**

$$\pi_{0,4} = g_{210} \circ g_{021} \circ g_{102} : \hat{H}_0^{in,1} \rightarrow \hat{H}_0^{in,1},$$

$$\pi_{0,4}(w, z_1, z_2)$$

$$= \left( z_1 z_2^{\frac{2+\varepsilon_x+\varepsilon_y}{1-\varepsilon_x}} w^{\frac{-3-3\varepsilon_x-3\varepsilon_y+\varepsilon_x\varepsilon_y}{1-\varepsilon_x^2}}, z_1^{-\frac{1+\varepsilon_y}{2}} z_2^{-\frac{3+\varepsilon_y^2}{2(1-\varepsilon_x)}} w^{\frac{9-3\varepsilon_x+3\varepsilon_y^2-\varepsilon_x\varepsilon_y^2}{2(1-\varepsilon_x^2)}}, z_1^{\frac{1-\varepsilon_x}{2}} z_2^{\frac{1+\varepsilon_y}{2}} w^{\frac{-1-\varepsilon_x-3\varepsilon_y+\varepsilon_x\varepsilon_y}{2(1+\varepsilon_x)}} \right)$$

$$\text{for } w^{\frac{2(2-\varepsilon_x-\varepsilon_y)}{(1+\varepsilon_x)(1-\varepsilon_y)}} < z_2 < w^{\frac{2}{1+\varepsilon_x}} \wedge z_1^{\frac{1+\varepsilon_y}{2}} > z_2^{-\frac{3+\varepsilon_y^2}{(1-\varepsilon_x)(1+\varepsilon_y)}} w^{\frac{9-3\varepsilon_x+3\varepsilon_y^2-\varepsilon_x\varepsilon_y^2}{(1-\varepsilon_x^2)(1+\varepsilon_y)}}$$

$$\pi_{1,4} = g_{021} \circ g_{102} \circ g_{210} : \hat{H}_1^{in,2} \rightarrow \hat{H}_1^{in,2},$$

$$\pi_{1,4}(w, z_1, z_2)$$

$$= \left( z_2 z_1^{\frac{-1-\varepsilon_x-3\varepsilon_y+\varepsilon_x\varepsilon_y}{1-\varepsilon_x^2}} w^{\frac{-1-\varepsilon_x-3\varepsilon_y+\varepsilon_x\varepsilon_y}{1-\varepsilon_x^2}}, z_1^{-\frac{2}{1-\varepsilon_x}} w^{-\frac{3+\varepsilon_x^2}{2(1-\varepsilon_x)}}, z_1^{\frac{2(2-\varepsilon_x-\varepsilon_y)}{1-\varepsilon_x^2}} w^{\frac{7-4\varepsilon_x-3\varepsilon_y+\varepsilon_x^2-\varepsilon_x^2\varepsilon_y}{2(1-\varepsilon_x^2)}} \right)$$

$$\text{for } 0 \leq z_2 < w^{\frac{1+\varepsilon_y}{2}} \wedge w^{\frac{-3+2\varepsilon_x+\varepsilon_x^2}{4}} < z_1 < w^{-\frac{7-4\varepsilon_x-3\varepsilon_y+\varepsilon_x^2-\varepsilon_x^2\varepsilon_y}{4(2-\varepsilon_x-\varepsilon_y)}}$$

$$\pi_{2,4} = g_{102} \circ g_{210} \circ g_{021} : \hat{H}_2^{in,0} \rightarrow \hat{H}_2^{in,0},$$

$$\pi_{2,4}(w, z_1, z_2)$$

$$= \left( z_1^{-\frac{3+\varepsilon_x^2}{2(1+\varepsilon_x)}} w^{-\frac{7+4\varepsilon_x+3\varepsilon_y+\varepsilon_x^2+\varepsilon_x^2\varepsilon_y}{2(1-\varepsilon_x^2)}}, z_2 z_1^{\frac{1-\varepsilon_x-3\varepsilon_y-\varepsilon_x\varepsilon_y}{2(1+\varepsilon_x)}} w^{\frac{(1+\varepsilon_y)(1-\varepsilon_x-3\varepsilon_y-\varepsilon_x\varepsilon_y)}{2(1-\varepsilon_x^2)}}, z_1^{\frac{2}{1+\varepsilon_x}} w^{\frac{2(2+\varepsilon_x+\varepsilon_y)}{1-\varepsilon_x^2}} \right)$$

$$\text{for } 0 \leq z_2 < \min \left\{ w^{\frac{1-\varepsilon_y}{1-\varepsilon_x}}, z_1^{\frac{1+\varepsilon_y}{2}} w^{\frac{3+\varepsilon_y^2}{2(1-\varepsilon_x)}} \right\} \wedge z_1 > w^{-\frac{7+4\varepsilon_x+3\varepsilon_y+\varepsilon_x^2+\varepsilon_x^2\varepsilon_y}{(3+\varepsilon_x^2)(1-\varepsilon_x)}}.$$

## C Transition matrices

In logarithmic coordinates  $\boldsymbol{\eta} = (\ln(w), \ln(z_1), \ln(z_2))$ , the maps  $g_{kjl}$  ( $j \neq k, l$ ) are linear and such that  $g_{kjl}(\boldsymbol{\eta}) = M_{kjl}\boldsymbol{\eta}$  where  $M_{kjl}$  are the transition matrices of the maps.

**$C_1$ -cycle:**

$$M_{212} = \begin{bmatrix} \frac{2}{1+\varepsilon_y} & 1 & 0 \\ \frac{1-\varepsilon_x}{1+\varepsilon_y} & 0 & 0 \\ -\frac{2}{1+\varepsilon_y} & 0 & 1 \end{bmatrix}, \quad M_{121} = \begin{bmatrix} -\frac{1-\varepsilon_y}{1-\varepsilon_x} & 0 & 1 \\ \frac{1+\varepsilon_x}{1-\varepsilon_x} & 1 & 0 \\ \frac{1+\varepsilon_y}{1-\varepsilon_x} & 0 & 0 \end{bmatrix}$$

$$M_1^{(1)} = M_{121}M_{212} = \begin{bmatrix} -\frac{2(2-\varepsilon_x-\varepsilon_y)}{(1-\varepsilon_x)(1+\varepsilon_y)} & -\frac{1-\varepsilon_y}{1-\varepsilon_x} & 1 \\ \frac{1+\varepsilon_x}{1-\varepsilon_x} & \frac{3+\varepsilon_x^2}{(1-\varepsilon_x)(1+\varepsilon_y)} & 0 \\ \frac{1+\varepsilon_y}{1-\varepsilon_x} & \frac{2}{1-\varepsilon_x} & 0 \end{bmatrix}$$

$$M_1^{(2)} = M_{212}M_{121} = \begin{bmatrix} \frac{-1+\varepsilon_x+3\varepsilon_y+\varepsilon_x\varepsilon_y}{(1-\varepsilon_x)(1+\varepsilon_y)} & 1 & \frac{2}{1+\varepsilon_y} \\ \frac{1-\varepsilon_x}{1+\varepsilon_y} & 0 & -\frac{1-\varepsilon_y}{1+\varepsilon_y} \\ \frac{3+\varepsilon_y^2}{(1-\varepsilon_x)(1+\varepsilon_y)} & 0 & -\frac{2}{1+\varepsilon_y} \end{bmatrix}$$

**$C_2$ -cycle:**

$$M_{202} = \begin{bmatrix} \frac{2}{1+\varepsilon_x} & 1 & 0 \\ \frac{1-\varepsilon_y}{1+\varepsilon_x} & 0 & 0 \\ -\frac{2}{1+\varepsilon_x} & 0 & 1 \end{bmatrix}, \quad M_{020} = \begin{bmatrix} -\frac{1-\varepsilon_x}{1-\varepsilon_y} & 0 & 1 \\ \frac{1+\varepsilon_y}{1-\varepsilon_y} & 1 & 0 \\ \frac{1+\varepsilon_x}{1-\varepsilon_y} & 0 & 0 \end{bmatrix}$$

$$M_2^{(0)} = M_{020}M_{202} = \begin{bmatrix} -\frac{2(2-\varepsilon_x-\varepsilon_y)}{(1+\varepsilon_x)(1-\varepsilon_y)} & -\frac{1-\varepsilon_x}{1-\varepsilon_y} & 1 \\ \frac{1+\varepsilon_y}{1-\varepsilon_y} & \frac{3+\varepsilon_y^2}{(1+\varepsilon_x)(1-\varepsilon_y)} & 0 \\ \frac{1+\varepsilon_x}{1-\varepsilon_y} & \frac{2}{1-\varepsilon_y} & 0 \end{bmatrix}$$

$$M_2^{(2)} = M_{202}M_{020} = \begin{bmatrix} \frac{-1+3\varepsilon_x+\varepsilon_y+\varepsilon_x\varepsilon_y}{(1+\varepsilon_x)(1-\varepsilon_y)} & 1 & \frac{2}{1+\varepsilon_x} \\ \frac{1-\varepsilon_y}{1+\varepsilon_x} & 0 & -\frac{1-\varepsilon_x}{1+\varepsilon_x} \\ \frac{3+\varepsilon_x^2}{(1+\varepsilon_x)(1-\varepsilon_y)} & 0 & -\frac{2}{1+\varepsilon_x} \end{bmatrix}$$

**$C_3$ -cycle:**

$$\begin{aligned}
M_{201} &= \begin{bmatrix} 1 & 1 & 0 \\ -\frac{1+\varepsilon_x}{2} & 0 & 1 \\ \frac{1-\varepsilon_y}{2} & 0 & 0 \end{bmatrix}, \quad M_{012} = \begin{bmatrix} -\frac{2}{1+\varepsilon_y} & 0 & 1 \\ \frac{1-\varepsilon_x}{1+\varepsilon_y} & 1 & 0 \\ \frac{2}{1+\varepsilon_y} & 0 & 0 \end{bmatrix}, \quad M_{120} = \begin{bmatrix} \frac{1+\varepsilon_x}{1-\varepsilon_y} & 1 & 0 \\ \frac{1+\varepsilon_y}{1-\varepsilon_y} & 0 & 0 \\ -\frac{1-\varepsilon_x}{1-\varepsilon_y} & 0 & 1 \end{bmatrix} \\
M_3^{(0)} = M_{120}M_{012}M_{201} &= \begin{bmatrix} \frac{-1-3\varepsilon_x-\varepsilon_y+\varepsilon_x\varepsilon_y}{1-\varepsilon_y^2} & \frac{-1-3\varepsilon_x-\varepsilon_y+\varepsilon_x\varepsilon_y}{1-\varepsilon_y^2} & 1 \\ -\frac{3+\varepsilon_y^2}{2(1-\varepsilon_y)} & -\frac{2}{1-\varepsilon_y} & 0 \\ \frac{7-3\varepsilon_x-4\varepsilon_y+\varepsilon_y^2-\varepsilon_x\varepsilon_y^2}{2(1-\varepsilon_y^2)} & \frac{2(2-\varepsilon_x-\varepsilon_y)}{1-\varepsilon_y^2} & 0 \end{bmatrix} \\
M_3^{(1)} = M_{201}M_{120}M_{012} &= \begin{bmatrix} \frac{-3-3\varepsilon_x-3\varepsilon_y+\varepsilon_x\varepsilon_y}{1-\varepsilon_y^2} & 1 & \frac{2+\varepsilon_x+\varepsilon_y}{1-\varepsilon_y} \\ \frac{9-3\varepsilon_y+3\varepsilon_x^2-\varepsilon_x^2\varepsilon_y}{2(1-\varepsilon_y^2)} & -\frac{1+\varepsilon_x}{2} & -\frac{3+\varepsilon_x^2}{2(1-\varepsilon_y)} \\ \frac{-1-3\varepsilon_x-\varepsilon_y+\varepsilon_x\varepsilon_y}{2(1+\varepsilon_y)} & \frac{1-\varepsilon_y}{2} & \frac{1+\varepsilon_x}{2} \end{bmatrix} \\
M_3^{(2)} = M_{012}M_{201}M_{120} &= \begin{bmatrix} -\frac{7+3\varepsilon_x+4\varepsilon_y+\varepsilon_y^2+\varepsilon_x\varepsilon_y^2}{2(1-\varepsilon_y^2)} & -\frac{3+\varepsilon_y^2}{2(1+\varepsilon_y)} & 0 \\ \frac{(1+\varepsilon_x)(1-3\varepsilon_x-\varepsilon_y-\varepsilon_x\varepsilon_y)}{2(1-\varepsilon_y^2)} & \frac{1-3\varepsilon_x-\varepsilon_y-\varepsilon_x\varepsilon_y}{2(1+\varepsilon_y)} & 1 \\ \frac{2(2+\varepsilon_x+\varepsilon_y)}{1-\varepsilon_y^2} & \frac{2}{1+\varepsilon_y} & 0 \end{bmatrix}
\end{aligned}$$

**$C_4$ -cycle:**

$$\begin{aligned}
M_{102} &= \begin{bmatrix} -\frac{2}{1+\varepsilon_x} & 0 & 1 \\ \frac{1-\varepsilon_y}{1+\varepsilon_x} & 1 & 0 \\ \frac{2}{1+\varepsilon_x} & 0 & 0 \end{bmatrix}, \quad M_{210} = \begin{bmatrix} 1 & 1 & 0 \\ -\frac{1+\varepsilon_y}{2} & 0 & 1 \\ \frac{1-\varepsilon_x}{2} & 0 & 0 \end{bmatrix}, \quad M_{021} = \begin{bmatrix} \frac{1+\varepsilon_y}{1-\varepsilon_x} & 1 & 0 \\ \frac{1+\varepsilon_x}{1-\varepsilon_x} & 0 & 0 \\ -\frac{1-\varepsilon_y}{1-\varepsilon_x} & 0 & 1 \end{bmatrix}
\end{aligned}$$

$$M_3^{(0)} = M_{210}M_{021}M_{102} = \begin{bmatrix} \frac{-3 - 3\varepsilon_x - 3\varepsilon_y + \varepsilon_x\varepsilon_y}{1 - \varepsilon_x^2} & 1 & \frac{2 + \varepsilon_x + \varepsilon_y}{1 - \varepsilon_x} \\ \frac{9 - 3\varepsilon_x + 3\varepsilon_y^2 - \varepsilon_x\varepsilon_y^2}{2(1 - \varepsilon_x^2)} & -\frac{1 + \varepsilon_y}{2} & -\frac{3 + \varepsilon_y^2}{2(1 - \varepsilon_x)} \\ \frac{-1 - \varepsilon_x - 3\varepsilon_y + \varepsilon_x\varepsilon_y}{2(1 + \varepsilon_x)} & \frac{1 - \varepsilon_x}{2} & \frac{1 + \varepsilon_y}{2} \end{bmatrix}$$

$$M_3^{(1)} = M_{201}M_{120}M_{012} = \begin{bmatrix} \frac{-1 - \varepsilon_x - 3\varepsilon_y + \varepsilon_x\varepsilon_y}{1 - \varepsilon_x^2} & \frac{-1 - \varepsilon_x - 3\varepsilon_y + \varepsilon_x\varepsilon_y}{1 - \varepsilon_x^2} & 1 \\ -\frac{3 + \varepsilon_x^2}{2(1 - \varepsilon_x)} & -\frac{2}{1 - \varepsilon_x} & 0 \\ \frac{7 - 4\varepsilon_x - 3\varepsilon_y + \varepsilon_x^2 - \varepsilon_x^2\varepsilon_y}{2(1 - \varepsilon_x^2)} & \frac{2(2 - \varepsilon_x - \varepsilon_y)}{1 - \varepsilon_x^2} & 0 \end{bmatrix}$$

$$M_3^{(2)} = M_{012}M_{201}M_{120} = \begin{bmatrix} -\frac{7 + 4\varepsilon_x + 3\varepsilon_y + \varepsilon_x^2 + \varepsilon_x^2\varepsilon_y}{2(1 - \varepsilon_x^2)} & -\frac{3 + \varepsilon_x^2}{2(1 + \varepsilon_x)} & 0 \\ \frac{(1 + \varepsilon_y)(1 - \varepsilon_x - 3\varepsilon_y - \varepsilon_x\varepsilon_y)}{2(1 - \varepsilon_x^2)} & \frac{1 - \varepsilon_x - 3\varepsilon_y - \varepsilon_x\varepsilon_y}{2(1 + \varepsilon_x)} & 1 \\ \frac{2(2 + \varepsilon_x + \varepsilon_y)}{1 - \varepsilon_x^2} & \frac{2}{1 + \varepsilon_x} & 0 \end{bmatrix}$$